The Orbital Motion in the LEO Region: Study of the 14:1 Resonance

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Introduction

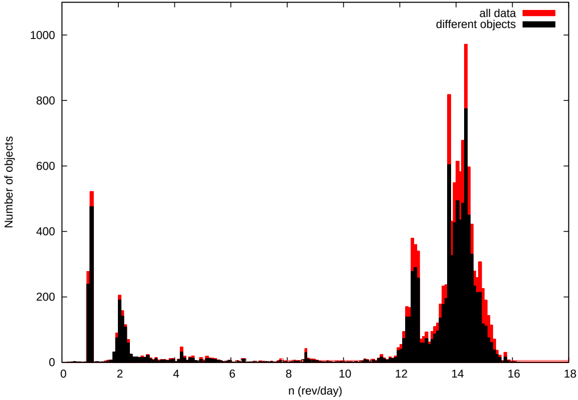
- The orbital motion of the catalogued objects can be analyzed using TLE data of the NORAD.
- The resonant period of the catalogued objects are analyzed observing the objects in deep resonance.
- In this work, a search about objects in resonant orbital motions is done.
- From the Earth gravitational potential and successive canonical transformations, the simplified Hamiltonian is defined.
- The 14:1 resonance is considered; in other words, the satellite completes fourteen revolutions while the Earth carries one.

Goals

- Define the resonance with the most of the objects in the LEO region.
- Define the equations of motion.
- Study the phase space and the time behavior of the semi-major axis and the angle ϕ_2.

Histogram of the mean motion

In the histogram of the mean motion is verified the most of the objects are $13 \leq n(rev/day) \leq 15$.



The criterium established is resonant period $Pres > 300$ days. $Pres$ is obtained by the relation,

$$Pres = \frac{2\pi}{\dot{\phi}_{lmpq}}. \quad (1)$$

and $\dot{\phi}_{lmpq}$ is calculated from.

$$\phi_{lmpq}(M, \omega, \Omega, \Theta) = (l - 2p + q)M + (l - 2p)\omega + m(\Omega - \Theta - \lambda_{lm}) + (l - m)\frac{\pi}{2} \ . \quad (2)$$

with Ω, ω, M are the classical keplerian elements; Θ is the Greenwich sidereal time and λ_{lm} is the corresponding reference longitude along the equator. So, $\dot{\phi}_{lmpq}$ is defined as

$$\dot{\phi}_{lmpq} = (l - 2p + q)\dot{M} + (l - 2p)\dot{\omega} + m(\dot{\Omega} - \dot{\Theta}) \ . \quad (3)$$

The terms $\dot{\omega}$, $\dot{\Omega}$ and $\dot{M}$ are given by

$$\dot{\omega} = -\frac{3}{4} J_2 n_o \left(\frac{a_e}{a_o}\right)^2 \frac{(1 - 5cos^2(I))}{(1 - e^2)^2}.$$

$$\dot{\Omega} = -\frac{3}{2} J_2 n_o \left(\frac{a_e}{a_o}\right)^2 \frac{(cos(I))}{(1 - e^2)^2}.$$

$$\dot{M} = n_o - \frac{3}{4} J_2 n_o \left(\frac{a_e}{a_o}\right)^2 \frac{(1 - 3cos^2(I))}{(1 - e^2)^{3/2}}. \quad (4)$$

a_e is the Earth mean equatorial radius, a_e=6378.140 *km*, J_2 is the second zonal harmonic, $J_2 = 1,0826 \times 10^{-3}$
The term $\dot{\Theta}$ in rad/day is

$$\dot{\Theta} = 1.00273790926 \times 2\pi. \quad (5)$$

The file analyzed is "*alldata_2011_045*", from the website of the Space Track and it corresponds to february 2011. See the Tab. 1 with more details about this file.

Table 1: 2-lines data of objects orbiting the Earth.

File	Number of data	Number of different objects
alldata_2011_045	15360	9745

The simulation considered objects with Resonant period > 300 days. See the Tab. 2 showing details about the simulation considered. The Tabs. 3 and 4 show the results by different regions defined by the value of the semi-major axis.

Table 2: Coefficients in the simulation.

coefficient k	coefficient q	coefficient m
$-50 \leq k \leq 50$	$-5 \leq q \leq 5$	$1 \leq m \leq 50$

Table 3: Number of different objects satisfying the condition of Pres > 300 days.

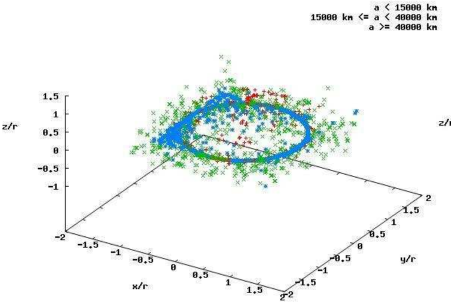
$a < 15000km$	$15000km \leq a < 40000km$	$a \geq 40000km$
1276	331	439

Table 4: Percentage of different objects satisfying the condition of Pres > 300 days.

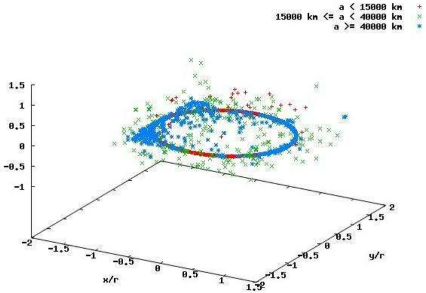
$a < 15000km$	$15000km \leq a < 40000km$	$a \geq 40000km$
62.37 %	16.18 %	21.46 %

The Fig. 2 show the space (x/r, y/r, z/r) around the Earth. a) with all catalogued objects; b) with only the resonant objects satisfying the condition P_{res}>300 days.

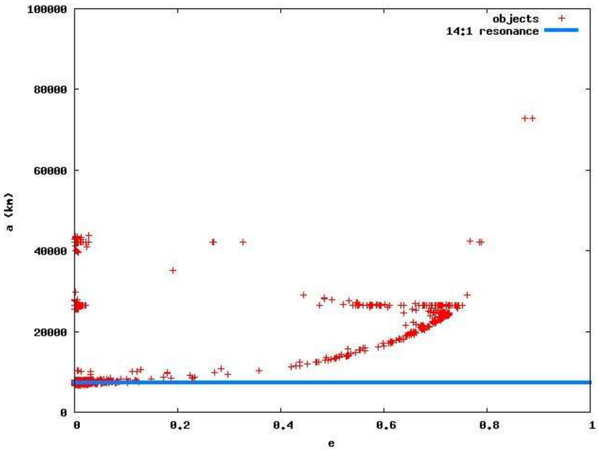
a)



b)



The Fig. 3 show the semi-major axis versus eccentricity, using only the data of the objects satisfying the condition of the *Pres* > 300 days.



The Fig. 4 show the semi-major axis versus inclination, using only the data of the objects satisfying the condition of the *Pres* > 300 days.

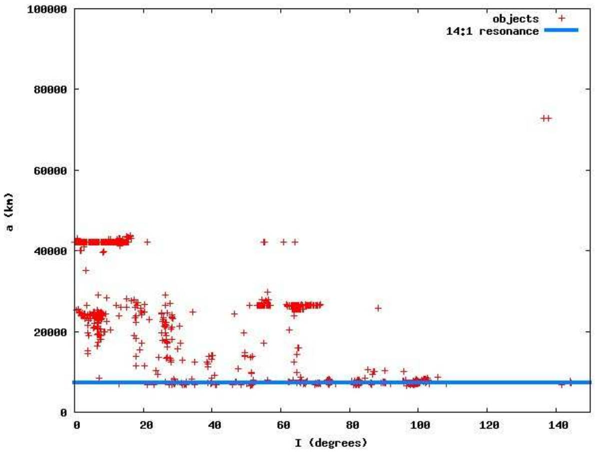


Table: Number of different objects by the number of "m", considering the value of semi-major axis up to 15000*km*

m	Number of objects	m	Number of objects
7	2	20	1
8	1	21	2
9	3	22	2
10	1	23	2
11	3	24	10
12	20	25	93
13	31	26	19
14	**351**	27	66
15	65	28	179
16	4	29	97
18	3	30	31

Gravitational Potential

$$V = \frac{\mu}{2a} + \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=+\infty}^{-\infty} \frac{\mu}{a} \left(\frac{a_e}{a} \right)^l J_{lm} F_{lmp}(I) G_{lpq}(e) cos(\phi_{lmpq}(M, \omega, \Omega, \Theta)), \quad (6)$$

μ is the Earth gravitational parameter, μ=3.986009 x 10^{14} m^3/s^2;
a, e, I, Ω, ω, M are the classical keplerian elements;
a_e is the Earth mean equatorial radius, a_e=6378.140 km;
J_{lm} is the spherical harmonic coefficient of degree l and order m;
$F_{lmp}(I)$ and $G_{lpq}(e)$ are Kaula's inclination and eccentricity functions, respectively.
Θ is the Greenwich sidereal time;
λ_{lm} is the corresponding reference longitude along the equator.
The argument $\phi_{lmpq}(M, \omega, \Omega, \Theta)$ is defined by the Eq.:

$$\phi_{lmpq}(M, \omega, \Omega, \Theta) = (l - 2p + q)M + (l - 2p)\omega + m(\Omega - \Theta - \lambda_{lm}) + (l - m)\frac{\pi}{2} .$$

where Θ is the Greenwich sidereal time and λ_{lm} is the corresponding reference longitude along the equator.

Canonical equations

In order to describe the problem in Hamiltonian form, Delaunay canonical variables are introduced

$$(a, e, I, \omega, \Omega, M) \rightarrow (L, G, H, l, g, h)$$

$$L = \sqrt{\mu a} \qquad G = \sqrt{\mu a(1 - e^2)} \qquad H = \sqrt{\mu a(1 - e^2)} cos(I)$$

$$l = M \qquad g = \omega \qquad h = \Omega$$

The resonance problem

Consider the resonance to be studied in this work; that is, the commensurability between the Earth's rotation angular velocity ω_e and the mean motion n. This commensurability can be expressed as

$$(l - 2p + q)n - m\omega_e \cong 0 \tag{7}$$

considering l, p, q and m as integers. The commensurability of the resonance studied, $l - 2p + q/m$, is defined by α. When this commensurability occurs, small divisors, associated to the tesseral harmonics, arise in the integration of the equations of motion. These terms are called resonant.

Simplified Hamiltonian

- Successive variables transformations in the Gravitational Potential.
- Hamiltonian = Secular terms + Resonant Terms
- The Hamiltonian has all the frequencies related to the α
    - $\alpha = l - 2p + q/m$, l-2p+q=1, m=14.

Final Dynamical System

The dynamical system generated by Hamiltonian $\hat{H}'_{2,f}$ is

$$\frac{dX_2}{dt} = -m\alpha \sum_{p=S}^{\infty} B_{2,(2p+k)mp(\alpha m-k)}(X_2, C_1, C_2) sen(\phi_{2,(2p+k)mp(\alpha m-k)}) \tag{8}$$

$$\begin{aligned} \frac{d\phi_2}{dt} &= m\alpha \frac{\mu^2}{X_2^3} - m\omega_e - m\alpha \sum_{j=1}^{\infty} \frac{\partial B_{2,2j,0,j,0}(X_2, C_1, C_2)}{\partial X_2} - \\ & -m\alpha \sum_{p=S}^{\infty} \frac{\partial B_{2,(2p+k)mp(\alpha m-k)}(X_2, C_1, C_2)}{\partial X_2} cos(\phi_{2,(2p+k)mp(\alpha m-k)}(x_2, \Theta_2)) \end{aligned} \tag{9}$$

The Eqns. (8) and (9) represent the motion equations in a resonance of commensurability α.

Final Dynamical System

where $\sum_{j=1}^{\infty} B_{2,2j,0,j,0}(X_2, C_1, C_2)$ represents the secular terms and

$$\sum_{p=s}^{\infty} B_{2,(2p+k)mp(\alpha m-k)}(X_2, C_1, C_2) cos\phi_{1,(2p+k)mp(\alpha m-k)}(x_2, \Theta_2)$$

represents the resonant terms with the same critical frequencies.
The angle $\phi_{2,(2p+k)mp(\alpha m-k)}(x_2, \Theta_2)$ is given by

$$\phi_{2,(2p+k)mp(\alpha m-k)}(x_2, \Theta_2) = \phi_2 - \phi_{2,(2p+k)mp(\alpha m-k),0} \;, \tag{10}$$

where $\phi_2 = m(\alpha x_2 - \Theta_2)$, and

$$\phi_{2,(2p+k)mp(\alpha m-k),0} = m\lambda_{(2p+k)m} - (2p+k-m)\frac{\pi}{2} = \phi_{1,lmp(\alpha m-k)0} \,. \tag{11}$$

Results

Figure 5 show the phase space *a* versus ϕ_2, and the Figs. 6 and 7 show the time behavior of the semi-major axis and ϕ_2 angle, according to the numerical integration of the motion equations, (8) and (9).

The initial values for inclinations is 87^0 and eccentricity is 0.019. The semi-major axis are around the critical semi-major axis.

Around the 14:1 resonance, there are several space debris orbiting the Earth, without control and risking the useful time of the artificial satellites in operation. The knowledge of regular or stable regions, in the LEO zone, can be very important to provide greater security for the orbital motion of artificial satellites and, possibly, lower fuel consumption with orbital maneuvers compared to unstable regions.

Fig. 5: a versus ϕ_2, considering the critical angle $\phi_{14146-1}$ associated to J_{1414}. The initial conditions for inclination and eccentricity are $I = 87^o$ and e=0.019, respectively.

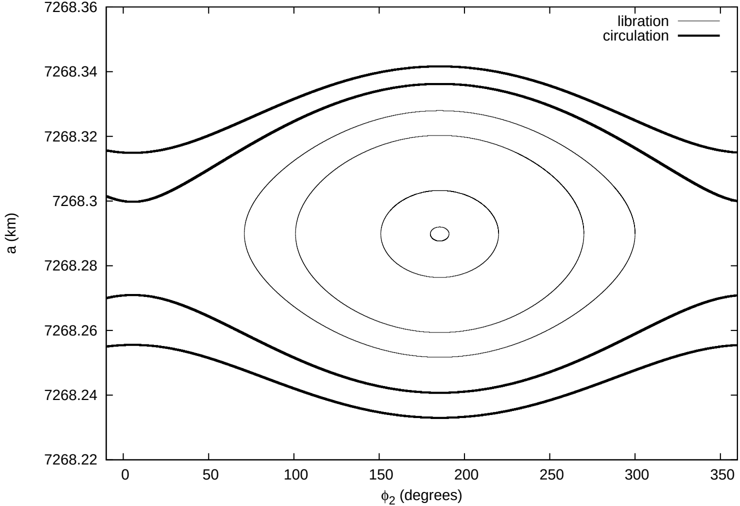


Fig. 6: Time behavior of the semi-major axis, considering the critical angle $\phi_{14146-1}$ associated to J_{1414}. The initial conditions for inclination and eccentricity are $I = 87^o$ and e=0.019, respectively.

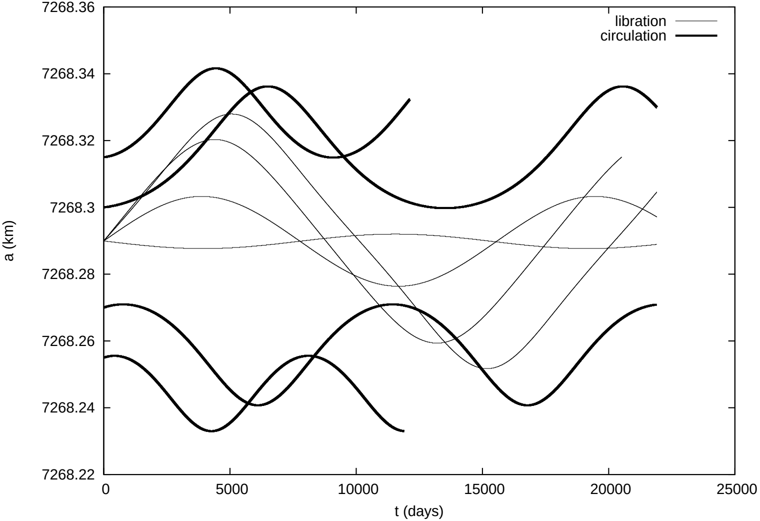
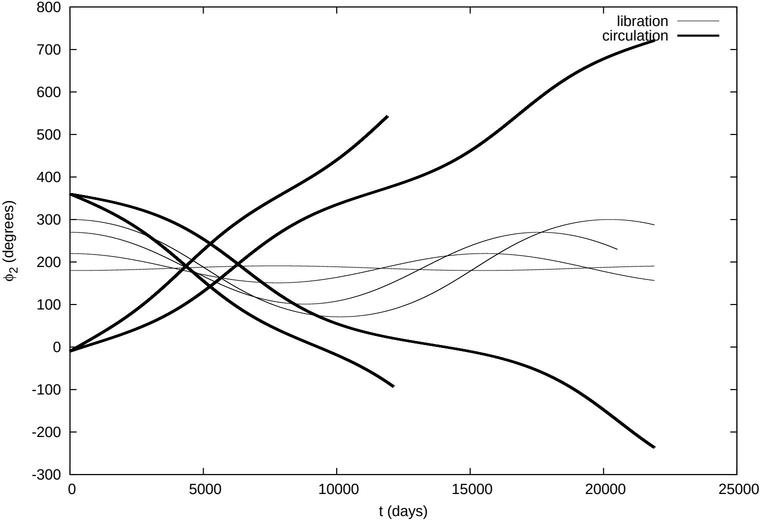


Fig. 7: Time behavior of the ϕ_2 angle, considering the critical angle $\phi_{14146-1}$ associated to J_{1414}. The initial conditions for inclination and eccentricity are $I = 87^o$ and e=0.019, respectively.



Conclusions

- In this work, the resonant objects around the Earth are studied.
- The most of the resonant objects are in the neighborhood of the 14:1 resonance.
- Considering this fact, the dynamical behavior of the critical angle $\phi_{14146-1}$ associated to the 14:1 resonance problem in the artificial satellites motion have been investigated from the simplified model.
- The results show the phase space, *a* versus ϕ_2, and the time behavior of the semi-major axis and ϕ_2 angle, considering the inclination, 87^0 and eccentricity, 0.019.
- Two different regions are observed in the numerical integration, libration and circulation regions.
- The theory developed for the resonant Hamiltonian and the equations of motion can be applied for any resonance.

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