The secular evolution of the orbit distance between two satellites of the Earth

Chiara Tardioli

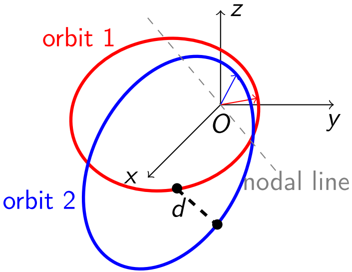
Department of Mathematics, University of Pisa

September 27, 2011

International Symposium on Orbit Propagation and Determination
Lille, France

Index

The Minimum Orbit Intersection Distance



The MOID is the minimum value of the 2-variable distance function d between a point on the first orbit and a point on the second one.
However, the two orbits can get close at other pairs of points, that correspond to local minima of d. So it is important to keep track of all these points.

Computation of the local minima

There are several papers in the literature on the computation of the minimal points of d, through the computation of all the critical points of d^2 (see e.g. Sitarski 1968).

Recently some algebraic methods to compute all the critical points of d^2 have been introduced:

- using Gröbner's basis (see Kholshevnikov and Vassiliev 1999);
- using the resultant theory (see Gronchi 2002, 2005).

They are both based on a polynomial formulation of the problem.

The orbit distance maps

- $\mathcal{E}$ is a vector of 10 components representing the geometrical configuration of the two orbits;
- V is a vector with components the two fast parameters along each orbit.

The critical points of $d^2(\mathcal{E}, \cdot)$ are the solutions of

$$\nabla_V d^2(\mathcal{E}, V) = 0\,.$$

We define the maps:

$$\begin{aligned} \mathcal{E} \mapsto d_h(\mathcal{E}) \quad &:= d(\mathcal{E}, V_h(\mathcal{E})) \\ \mathcal{E} \mapsto d_{min}(\mathcal{E}) &:= \min_h d_h(\mathcal{E}) \end{aligned}$$

giving the values of d at its local minima of indices h and its absolute minimum, respectively.

Remark: For fixed $\mathcal{E}$, the map d_{min} corresponds to the MOID.

The orbit distance maps

- $\mathcal{E}$ is a vector of 10 components representing the geometrical configuration of the two orbits: $\mathcal{E} = (a_1, e_1, \mathrm{I}_1, \Omega_1, \omega_1; a_2, e_2, \mathrm{I}_2, \Omega_2, \omega_2)$;
- V is a vector with components the two fast parameters along each orbit: $V = (v_1, v_2)$.

The critical points of $d^2(\mathcal{E}, \cdot)$ are the solutions of

$$\nabla_V d^2(\mathcal{E}, V) = 0\,.$$

We define the maps:

$$\begin{aligned} \mathcal{E} &\mapsto d_h(\mathcal{E}) &&:= d(\mathcal{E}, V_h(\mathcal{E})) \\ \mathcal{E} &\mapsto d_{min}(\mathcal{E}) &&:= \min_h d_h(\mathcal{E}) \end{aligned}$$

giving the values of d at its local minima of indices h and its absolute minimum, respectively.

Remark: For fixed $\mathcal{E}$, the map d_{min} corresponds to the MOID.

Problems in computing the uncertainty of d_h and d_{min}

The observational errors produce an uncertainty in the computed orbits and in their distance values.
To compute the uncertainty of d_h (for d_{min} is the same) we can use the standard covariance propagation formulae

$$\Gamma_{d_h} = \left[\frac{\partial d_h}{\partial \mathcal{E}}(\mathcal{E})\right] \Gamma_{\mathcal{E}} \left[\frac{\partial d_h}{\partial \mathcal{E}}(\mathcal{E})\right]^T . \quad (1)$$

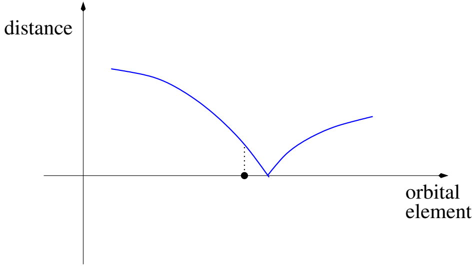
Problems:

- the derivative $\frac{\partial d_h}{\partial \mathcal{E}}$ does not exist at orbit crossings, i.e. when $d_h = 0$.
- the confidence interval may allow negative values of the distance d_h when the uncertainty is very small, while $d_h > 0$.

We will use the results in (Gronchi and Tommei 2007) to overcome these problems and give a meaningful uncertainty to d_h even in these cases.

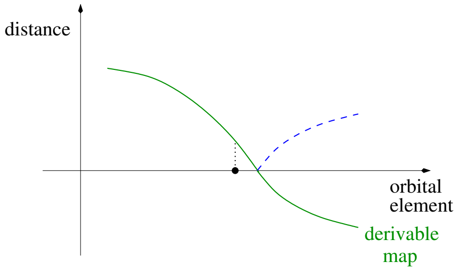
Idea

Let us vary only one orbital element



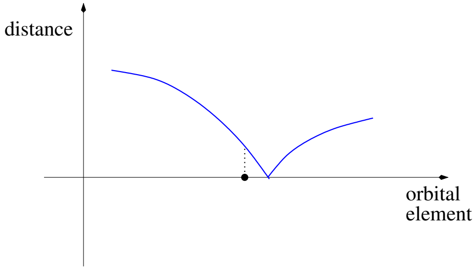
Idea

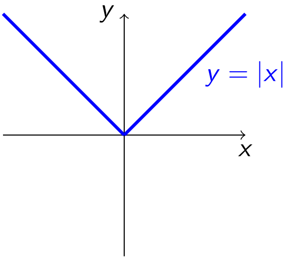
Let us vary only one orbital element



Idea

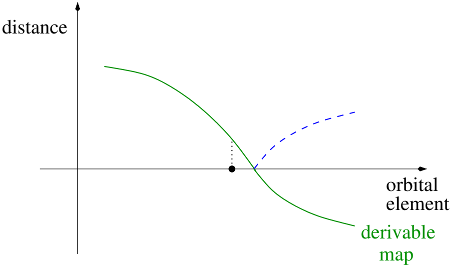
Let us vary only one orbital element

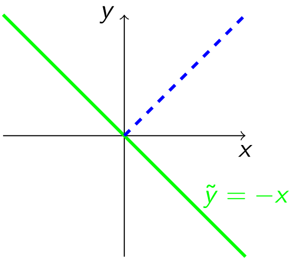




Idea

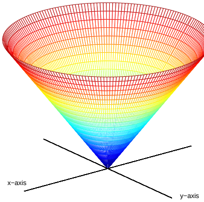
Let us vary only one orbital element





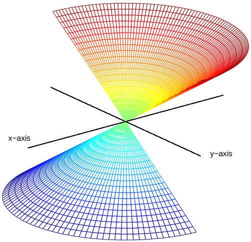
Hint: Changing the sign to d_h, we obtain a differentiable function even at orbit crossings.

For a function of several variables, the search for a smoothing through a change of sign can be difficult.



$$f(x,y) = \sqrt{x^2+y^2}$$

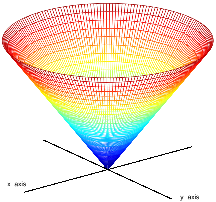
f is not differentiable in the origin.

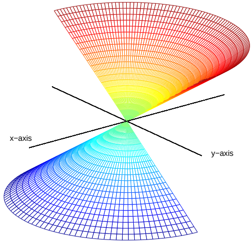


$$\tilde{f}(x,y) = \begin{cases} -f(x,y) & \text{per } x > 0 \\ f(x,y) & \text{per } x < 0 \end{cases}$$

Change of sign and restriction of the domain

For a function of several variables, the search for a smoothing through a change of sign can be difficult.





$$f(x,y) = \sqrt{x^2+y^2}$$

$$\tilde{f}(x,y) = \begin{cases} -f(x,y) & \text{per } x > 0 \\ f(x,y) & \text{per } x < 0 \end{cases}$$

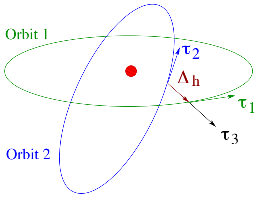
f is not differentiable in the origin.

$\tilde{f}$ can be extended with differentiability in the origin.

Giving a sign to d_h

We give a simple geometric interpretation of the choice of the sign for d_h.

- τ_1, τ_2 are the tangent vectors to the orbits at their mutually closest points.
- $\tau_3 = \tau_1 \times \tau_2$.
- Δ_h is the vector joining the two orbits on these points: $\Delta_h \perp \tau_1, \tau_2$; $\Delta_h || \tau_3$.



$$\tilde{d}_h = \begin{cases} +d_h & \text{if } \Delta_h = \tau_3 \\ -d_h & \text{if } \Delta_h = -\tau_3 \end{cases}$$

We can extend $\tilde{d}_h$ at orbit crossings, i.e. where $\tilde{d}_h = 0$.

Remark: We can not extend $\tilde{d}_h$ at configurations in which τ_1 and τ_2 are parallel.

The satellite dynamical model

We consider only the main gravitational perturbation due to the non-spherical shape of the Earth.
Then, by the averaging principle, the secular evolution of a satellite is given by

$$
\begin{aligned}
\bar{a} &= a_o\,, \quad \bar{e} = e_o\,, \quad \bar{\mathrm{I}} = \mathrm{I}_o\,,\\
\bar{\Omega} &= \Omega_o - \frac{3}{2}\,\frac{J_2 \mathrm{R}_\oplus^2}{p^2}\,\bar{n}\,(\cos I)\,t\,,\\
\bar{\omega} &= \omega_o + \frac{3}{2}\,\frac{J_2 \mathrm{R}_\oplus^2}{p^2}\,\bar{n}\left(2 - \frac{5}{2}\sin^2 I\right)t\,,
\end{aligned}
\tag{2}
$$

where

$(a_o, e_o, \mathrm{I}_o, \Omega_o, \omega_o)$ are the initial Keplerian orbital elements;

J_2 is the second zonal harmonic in the Earth gravitational field;

$\mathrm{R}_\oplus$ is the Earth equatorial radius;

$p = a_o(1 - e_o^2)$ is the conic parameter;

$\bar{n}$ is the averaged mean motion.

The map $\widetilde{d}_h$ and its uncertainty

The secular evolution of the orbit distance (with sign) is defined by

$$\widetilde{d}_h(t) = \tilde{d}_h(\bar{\mathcal{E}}(t)) \tag{3}$$

where $\bar{\mathcal{E}}(t)$ is the 10-vector giving the evolution of the two trajectories. For a fixed time t, its uncertainty is given by the propagation formulae:

$$\Gamma_{\widetilde{d}_h} = \left[\frac{\partial \widetilde{d}_h}{\partial \mathcal{E}}(\bar{\mathcal{E}})\right] \Gamma_{\bar{\mathcal{E}}} \left[\frac{\partial \widetilde{d}_h}{\partial \mathcal{E}}(\bar{\mathcal{E}})\right]^T \tag{4}$$

with $\Gamma_{\bar{\mathcal{E}}}$ the 10×10 covariance submatrix

$$\Gamma_{\bar{\mathcal{E}}} = \frac{\partial \bar{\mathcal{E}}}{\partial \mathcal{E}_o} \Gamma_{\mathcal{E}_o} \left[\frac{\partial \bar{\mathcal{E}}}{\partial \mathcal{E}_o}\right]^T \tag{5}$$

where $\mathcal{E}_o$ is the initial condition for the secular evolution and $\Gamma_{\mathcal{E}_o}$ its 10×10 covariance submatrix.

Remark: In the space debris case, this uncertainty is typically small.

Test I: the secular evolution of the orbital elements

	orbit 1	orbit 2
a_o	7866.0	7864.9
e_o	0.0038	0.0021
I_o	73.546	102.069
Ω_o	147.449	205.237
ω_o	1.549	132.325

The initial orbital elements and covariance matrices have been created by a new orbit determination method based on the first integrals of the Kepler problem (see Gronchi et al. 2010).

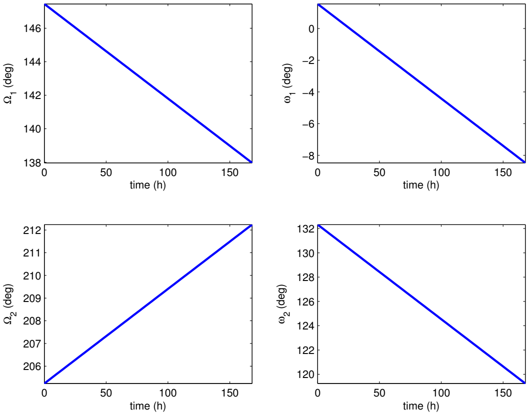


Figure: Secular evolution of angles $\bar{\Omega}_1, \bar{\omega}_1, \bar{\Omega}_2, \bar{\omega}_2$ for a two orbit configuration with initial orbital elements in the table on the left.

Test I : the secular evolution of $\widetilde{d}_1, \widetilde{d}_2$

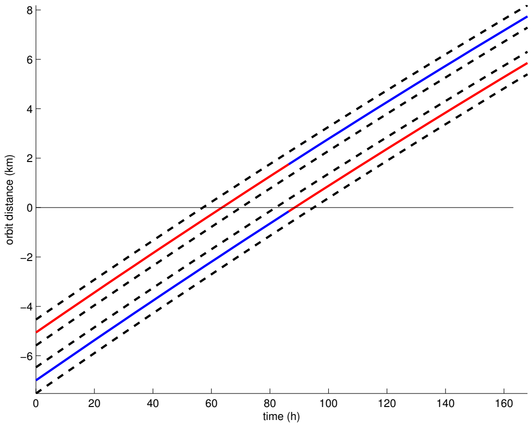


Figure: The red line is $\widetilde{d}_{min}$, the blue line is the second minimum value of d. The two continuous lines are $\widetilde{d}_1$ and $\widetilde{d}_2$, respectively above and below. The dashed lines represent the uncertainty bounds.

Remark: In this example $\widetilde{d}_1$ and $\widetilde{d}_2$ exchange their role as absolute minimum.

Test II : interval of possible crossing times

We perform an interpolation with splines at black nodes of the uncertainty bounds.

	orbit 1
a_o	7915.8
e_o	0.0146
I_o	73.912
Ω_o	321.864
ω_o	316.498

	orbit 2
a_o	7853.2
e_o	0.0064
I_o	102.379
Ω_o	181.840
ω_o	343.026

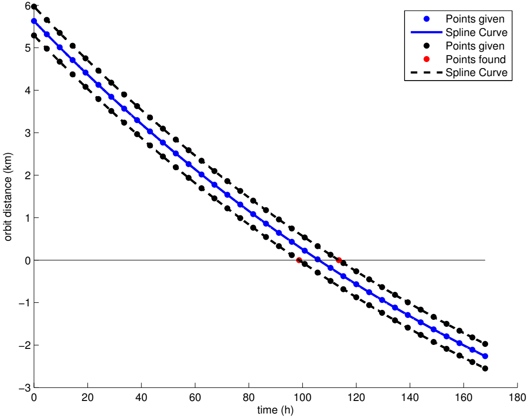
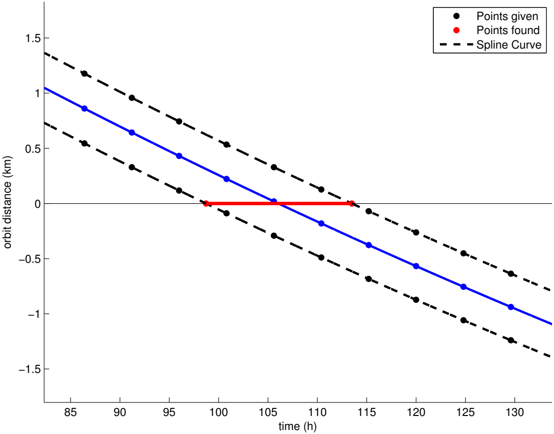


Figure: The blue line represents the secular evolution of $\widetilde{d}_1$, the black broken lines are the uncertainty bounds.

We perform an interpolation with splines at black nodes of the uncertainty bounds.

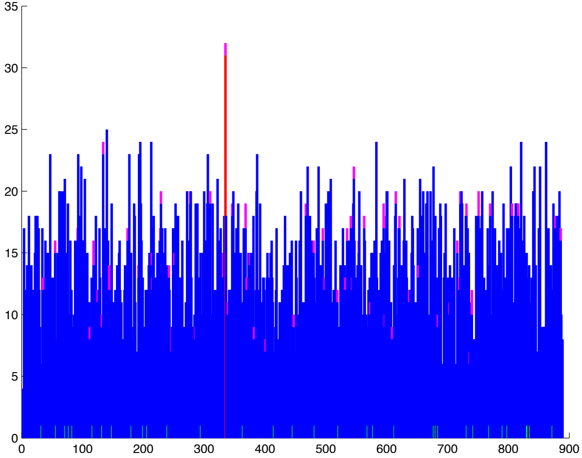


The red segment is the interval of crossing times $[t_a, t_b]$:

$$t_a = 98.7254 \text{ h}$$
$$t_b = 113.4934 \text{ h}$$
$$t^{nom} = 106.0461 \text{ h}$$
$$\Delta t = 14.7679 \text{ h}$$

Test III : a LEO population

We have also performed a test with a catalog of 874 orbits (Dimare et al. 2011) with $a \leqslant 8600$ km and 5 km $\leqslant |\widetilde{d}_{min}| \leqslant$ 10 km at the initial time. We detected 4689 orbit crossings in a week time span.



- the maximum crossing number is 31 (red line);
- 32 orbits have only one orbit crossing (green lines);
- there are 29 orbit couples whose trajectories intersect twice and 8 three times (magenta lines).

Conclusions

In this simplified dynamical model the secular evolution of an artificial satellite is always computable explicitly and it is a differentiable map of initial orbital elements.

⇓

We can always perform the computation of the distance between two confocal orbits together with its uncertainty for a suitable time span. Then, we may use these results to define an interval of possible crossing times.

⇓

If the catalog of space debris contains N orbits we have $N(N-1)/2$ couples to examine. We may select those pairs with a not empty crossing interval and use them as input orbit couples for collision avoidance algorithms.

Conclusions

In this simplified dynamical model the secular evolution of an artificial satellite is always computable explicitly and it is a differentiable map of initial orbital elements.

We can always perform the computation of the distance between two confocal orbits together with its uncertainty for a suitable time span. Then, we may use these results to define an interval of possible crossing times.

If the catalog of space debris contains N orbits we have $N(N-1)/2$ couples to examine. We may select those pairs with a not empty crossing interval and use them as input orbit couples for collision avoidance algorithms.

Conclusions

In this simplified dynamical model the secular evolution of an artificial satellite is always computable explicitly and it is a differentiable map of initial orbital elements.

⬇

We can always perform the computation of the distance between two confocal orbits together with its uncertainty for a suitable time span. Then, we may use these results to define an interval of possible crossing times.

⬇

If the catalog of space debris contains N orbits we have $N(N-1)/2$ couples to examine. We may select those pairs with a not empty crossing interval and use them as input orbit couples for collision avoidance algorithms.

References

K.V. Kholshevnikov and N. Vassiliev : 1999, *On the distance function between two Keplerian elliptic orbits*, Cel. Mech. Dyn. Astron. 75, pp 75-83.

G. F. Gronchi : 2005, *An algebraic method to compute the critical points of the distance function between two Keplerian orbits*, Cel. Mech. Dyn. Ast. 93, pp 295-329.

G. F. Gronchi and G. Tommei : 2007, *On the uncertainty of the minimal distance between two confocal keplerian orbits*, Dis. Con. Dyn. Sys.-Series B Vol. 7, No. 4, pp 755-778.

G. F. Gronchi and A. Milani : 1998, *Averaging on Earth-crossing orbits*, Cel. Mech. Dyn. Ast. 71/2, pp. 109-136.

A. E. Roy : 1988, *Orbital Motion*, 3rd Ed.

G. F. Gronchi, L. Dimare and A. Milani : 2010, *Orbit determination with the two-body integrals*, , Cel. Mech. Dyn. Ast. 107, pp. 299-318.

L. Dimare, D. Farnocchia, G.F. Gronchi, A. Milani, F. Bernardi : 2011, *Innovative system of very wide field optical sensors for space surveillance in the LEO region*, Proceedings of the 12th AMOS conference, 13-16 September 2011, Wailea, Hawaii.