Orbit propagation with Lie-Deprit methods in satellite dynamics

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The tesseral model of the satellite

Terms of the Hamiltonian

$$\mathcal{H}_k = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r},$$

$$\mathcal{H}_c = -\omega N,$$

$$\mathcal{H}_m = \frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2 J_2 P_2(\sin i \sin\theta),$$

$$\mathcal{H}_{zt} = \mathcal{H}_2^t + \sum_{k=3}^{n} \mathcal{H}_k^t,$$

$$\mathcal{H}_2^t = -\frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2 \sum_{j=1}^{2} \left(C_{2,j}\cos j\lambda + S_{2,j}\sin j\lambda\right) P_{2,j}(\sin i \sin\theta),$$

$$\mathcal{H}_k^t = -\frac{\mu}{r}\left(\frac{\alpha}{r}\right)^k \sum_{j=0}^{k} \left(C_{k,j}\cos j\lambda + S_{k,j}\sin j\lambda\right) P_{k,j}(\sin i \sin\theta).$$

Scaling the Hamiltonian

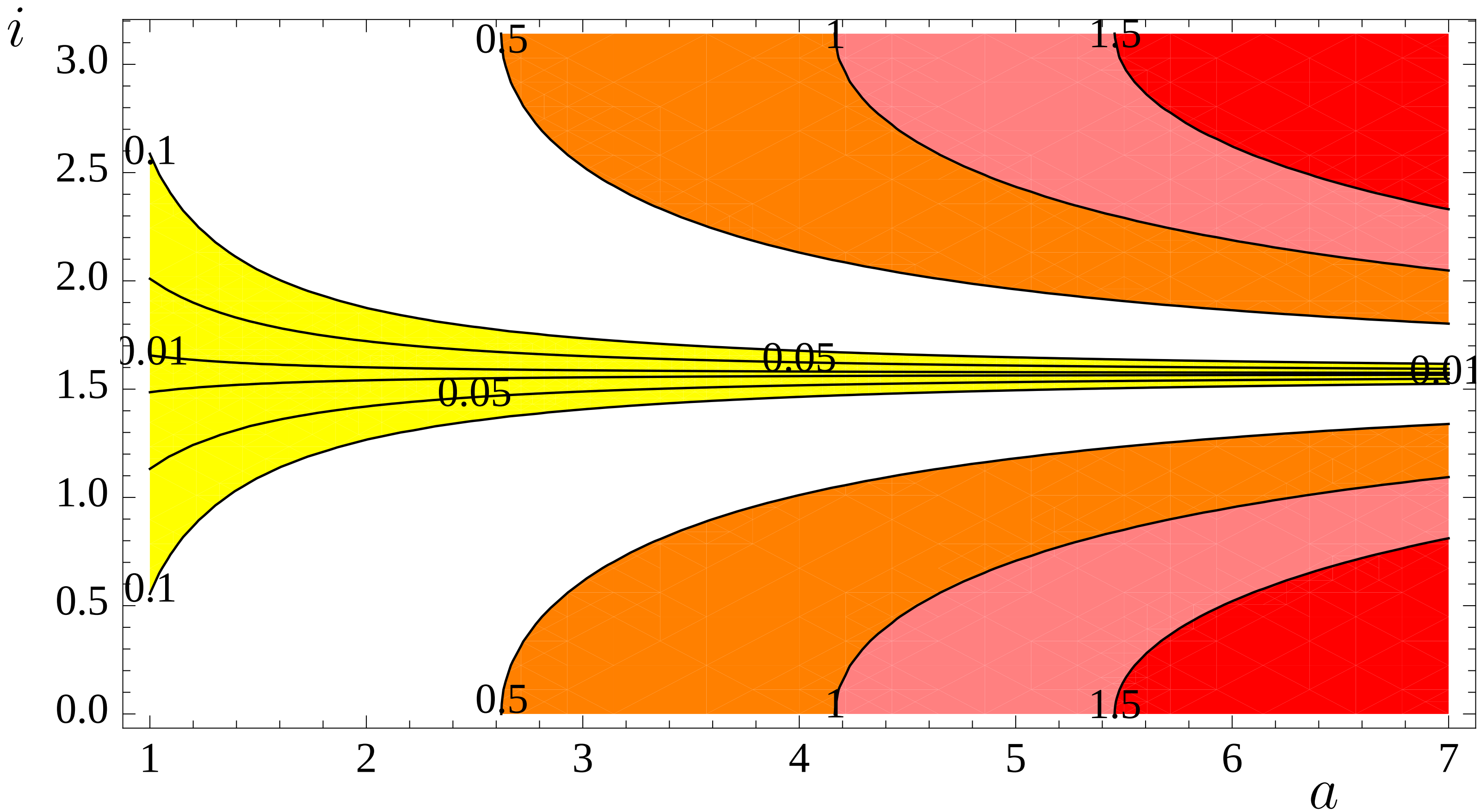
$$\mathcal{H} = \mathcal{H}_0 + \epsilon \mathcal{H}_1 + \frac{\epsilon^2}{2!}\mathcal{H}_2 + \frac{\epsilon^4}{4!}\mathcal{H}_4,$$

$$\begin{aligned}
\mathcal{H}_0 &= \mathcal{H}_k, \\
\mathcal{H}_1 &= \frac{\mathcal{H}_c}{\epsilon}, \qquad \epsilon \approx \frac{\mathcal{H}_c}{\mathcal{H}_k}, \qquad \epsilon? \\
\mathcal{H}_2 &= \frac{2!\,\mathcal{H}_m}{\epsilon^2}, \\
\mathcal{H}_4 &= \frac{4!\,\mathcal{H}_{zt}}{\epsilon^4}.
\end{aligned}$$

Validity of the Hamiltonian ordering (Earth)

Level curves of

$$\epsilon = |\mathcal{H}_c|/|\mathcal{H}_k| = (-\omega\sqrt{\mu a(1-e^2)}\cos i)/(-\mu/2a)$$



Lie-Deprit method

Lie Transform

$$\mathcal{H}(\boldsymbol{x}, \boldsymbol{X}; \epsilon) = \sum_{n \geq 0} \frac{\epsilon^n}{n!} \mathcal{H}_n(\boldsymbol{x}, \boldsymbol{X})$$

$$\downarrow$$

$$W(\boldsymbol{x}, \boldsymbol{X}; \epsilon) = \sum_{n \geq 0} \frac{\epsilon^n}{n!} W_n(\boldsymbol{x}, \boldsymbol{X})$$

$$\downarrow$$

$$\mathcal{K}(\boldsymbol{y}, \boldsymbol{Y}; \epsilon) = \sum_{n \geq 0} \frac{\epsilon^n}{n!} \mathcal{K}_n(\boldsymbol{y}, \boldsymbol{Y})$$

Lie triangle

$$\mathcal{H}_{i,j} = \mathcal{H}_{i+1,j-1} + \sum_{k=0}^{i} \binom{i}{k} (\mathcal{H}_{i-k,j-1}, W_{k+1}) \, .$$

$$\begin{array}{ccccccc}
 & & & \mathcal{H}_{0,0} & & & \\
 & & \swarrow & & & & \\
 & & \mathcal{H}_{1,0} & \longrightarrow & \mathcal{H}_{0,1} & & \\
 & \swarrow & & \swarrow & & & \\
 & \mathcal{H}_{2,0} & \longrightarrow & \mathcal{H}_{1,1} & \longrightarrow & \mathcal{H}_{0,2} & \\
\swarrow & & \swarrow & & \swarrow & & \\
\mathcal{H}_{3,0} & \longrightarrow & \mathcal{H}_{2,1} & \longrightarrow & \mathcal{H}_{1,2} & \longrightarrow & \mathcal{H}_{0,3} \\
\swarrow & \swarrow & & \swarrow & & \swarrow &
\end{array}$$

Lie triangle

$$\mathcal{H}_{i,j} = \mathcal{H}_{i+1,j-1} + \sum_{k=0}^{i} \binom{i}{k} (\mathcal{H}_{i-k,j-1}, W_{k+1}) .$$

$$\mathcal{H}_{n,0} = \mathcal{H}_n, \quad \longrightarrow \quad \mathcal{H}_{0,n} = \mathcal{K}_n.$$

$$\begin{array}{lllllllll}
\mathcal{H}_{n-1,1} & = & \mathcal{H}_{n,0} & +\dots+ & k_{12}(\mathcal{H}_{20}, W_{n-2}) & + & k_{11}(\mathcal{H}_{10}, W_{n-1}) & + & (\mathcal{H}_{00}, W_n) \\
\mathcal{H}_{n-2,2} & = & \mathcal{H}_{n-1,1} & +\dots+ & k_{22}(\mathcal{H}_{11}, W_{n-2}) & + & (\mathcal{H}_{01}, W_{n-1}) & & \\
\mathcal{H}_{n-3,3} & = & \mathcal{H}_{n-2,2} & +\dots+ & (\mathcal{H}_{02}, W_{n-2}) & & & & \\
\dots & = & \dots & & & & & & \\
\mathcal{H}_{0,n} & = & \dots & & & & & &
\end{array}$$

$$k_{12} = \binom{n-1}{n-3} = \frac{1}{2}(n-1)(n-2), \quad k_{11} = \binom{n-1}{n-2} = (n-1), \quad k_{22} = \binom{n-2}{n-3} = (n-2).$$

Homological equation

Tilde–normal elements relation:

$$\mathcal{H}_{i,j} = \widetilde{\mathcal{H}}_{i,j} + (\mathcal{H}_{00}, W_n) = \widetilde{\mathcal{H}}_{i,j} - \mathcal{L}_0 W_n,\ i < n$$

Order n: tilde elements: (Lie triangle with $W_n = 0$)

$$\begin{array}{lclclcl}
\widetilde{\mathcal{H}}_{n,0} & = & \mathcal{H}_{n,0} & & & & \\
\widetilde{\mathcal{H}}_{n-1,1} & = & \widetilde{\mathcal{H}}_{n,0} & +\ldots+ & k_{12}(\mathcal{H}_{20}, W_{n-2}) & + & k_{11}(\mathcal{H}_{10}, W_{n-1}), \\
\widetilde{\mathcal{H}}_{n-2,2} & = & \widetilde{\mathcal{H}}_{n-1,1} & +\ldots+ & k_{22}(\widetilde{\mathcal{H}}_{11}, W_{n-2}) & + & (\widetilde{\mathcal{H}}_{01}, W_{n-1}), \\
\widetilde{\mathcal{H}}_{n-3,3} & = & \widetilde{\mathcal{H}}_{n-2,2} & +\ldots+ & (\widetilde{\mathcal{H}}_{02}, W_{n-2}), & & \\
\ldots & = & \ldots & & & & \\
\widetilde{\mathcal{H}}_{0,n} & = & \ldots & & & &
\end{array}$$

Homological equation

$$\boxed{\widetilde{\mathcal{H}}_{0,n} = \mathcal{H}_{0,n} + \mathcal{L}_0 W_n}$$

Lie-Deprit method

$$\boxed{\widetilde{\mathcal{H}}_{0,n} = \mathcal{H}_{0,n} + \mathcal{L}_0 W_n}$$

1. Compute $\widetilde{\mathcal{H}}_{0,n}$
2. Split $\widetilde{\mathcal{H}}_{0,n}$ in two parts: $\widetilde{\mathcal{H}}_{0,n} = \mathcal{H}^{(1)}_{0,n} + \mathcal{H}^{(2)}_{0,n}$.
3. Choose the new Hamiltonian $\mathcal{H}_{0,n} = \mathcal{H}^{(1)}_{0,n}$.
4. Obtain W_n as an integral of the PDE $\mathcal{L}_0 W_n = \mathcal{H}^{(2)}_{0,n}$.
5. Complete the elements $\mathcal{H}_{i,j}$ adding $\mathcal{L}_0 W_n$ to the tilde elements.

Elimination of the parallax

$$\boxed{\mathcal{L}_0 W_n + \mathcal{H}_{0,n} = \widetilde{\mathcal{H}}_{0,n}}$$

$$\mathcal{L}_0 \left[\sum_{j \geq 0} \frac{1}{j} \left(C_j \sin j\theta - S_j \cos j\theta \right) \right] + \frac{\Theta}{r^2} C_0 = \frac{\Theta}{r^2} F,$$

$$F = \sum_{j \geq 0} \left(C_j \cos j\theta + S_j \sin j\theta \right), \quad C_j, S_j \in ker(\mathcal{L}_0)$$

In the tesseral problem the node ν appears in the expression of $\widetilde{\mathcal{H}}_{0,n}$.

Delaunay Normalization

$$\mathcal{H}_K = -\frac{\mu^2}{2L^2}, \quad \mathcal{L}_0 = \frac{\mu^2}{L^3}\frac{\partial}{\partial \ell}$$

The variables r and f expanded in powers of the eccentricity. Only order three.

$$\boxed{\mathcal{L}_0 W_n + \mathcal{H}_{0,n} = \widetilde{\mathcal{H}}_{0,n}}$$

$$\mathcal{H}_{0,n} = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{\mathcal{H}}_{0,n}\, d\ell, \quad W_n = \frac{L^3}{\mu^2}\int \left(\widetilde{\mathcal{H}}_{0,n} - \mathcal{H}_{0,n}\right) d\ell.$$

Extended normalization (eliminate g and h)

$$\mathcal{H}_{0,0} = \mathcal{H}_{0,0}(L), \quad \mathcal{H}_{1,0} = \mathcal{H}_{1,0}(H), \quad \mathcal{H}_{2,0} = \mathcal{H}_{2,0}(L, G, H)$$

$$W_n = W_n^\ell(\ell, g, h, L, G, H) + W_n^h(\_, g, h, L, G, H) + W_n^g(\_, g, \_, L, G, H)$$

$$\mathcal{L}_0 W_n = \mathcal{L}_0 W_n^l, \quad \mathcal{L}_1 W_n = \mathcal{L}_1 W_n^h, \quad \mathcal{L}_2 W_n = \mathcal{L}_2 W_n^g.$$

$$\begin{array}{lllllll}
\mathcal{H}_{n-1,1} & = & \mathcal{H}_{n,0} & +\dots+ & k_{12}(\mathcal{H}_{20}, W_{n-2}) & + & k_{11}(\mathcal{H}_{10}, W_{n-1}) & + & (\mathcal{H}_{00}, W_n) \\
\mathcal{H}_{n-2,2} & = & \mathcal{H}_{n-1,1} & +\dots+ & k_{22}(\mathcal{H}_{11}, W_{n-2}) & + & (\mathcal{H}_{01}, W_{n-1}) \\
\mathcal{H}_{n-3,3} & = & \mathcal{H}_{n-2,2} & +\dots+ & (\mathcal{H}_{02}, W_{n-2}) \\
\dots & = & \dots
\end{array}$$

Normalization begins at order three: $\mathcal{H}_{01} = \mathcal{H}_{10}, \quad \mathcal{H}_{02} = \mathcal{H}_{20}$

Homological equation:

$$\widetilde{\mathcal{H}}_{n0} = \mathcal{H}_{0n} + \frac{1}{2} n(n-1)\mathcal{L}_2 W_{n-2}^g + n\mathcal{L}_1 W_{n-1}^h + \mathcal{L}_0 W_n^l.$$

$$\begin{aligned}
\widetilde{\mathcal{H}}_{n0} \quad = \quad & \widetilde{\mathcal{H}}_{n0}^{\emptyset}(_, _, _, L, G, H) + \widetilde{\mathcal{H}}_{n0}^g(_, g, _, L, G, H) + \\
& \widetilde{\mathcal{H}}_{n0}^h(_, g, h, L, G, H) + \widetilde{\mathcal{H}}_{n0}^l(\ell, g, h, L, G, H)
\end{aligned}$$

Extended normalization (eliminate g and h)

$$\mathcal{H}_{00} = -\frac{\mu^2}{2L^2}, \quad \mathcal{H}_{10} = -\omega H, \quad \mathcal{H}_{20} = \mathcal{H}_{20}(L, G, H),$$

Extended normalization.

$$\begin{aligned}
\mathcal{L}_0 W_n^\ell &= \frac{\mu^2}{L^3}\frac{\partial W_n^\ell}{\partial \ell}, & W_n^\ell &= \frac{L^3}{\mu^2}\int \left(\mathcal{L}_0 W_n^\ell\right) d\ell \\
\mathcal{L}_1 W_n^h &= -\omega\frac{\partial W_n^h}{\partial h}, & W_n^h &= -\frac{1}{\omega}\int \left(\mathcal{L}_0 W_n^h\right) dh \\
\mathcal{L}_2 W_n^g &= G^*\frac{\partial W_n^g}{\partial g}, & W_n^g &= -\frac{4G^4L^3}{3Ki_c}\int \left(\mathcal{L}_0 W_n^g\right) dg
\end{aligned}$$

Numerical results

Plots meaning

- ☐ Analytical versus numerical (one month) propagation.
- ☐ Horizontal axis: a.

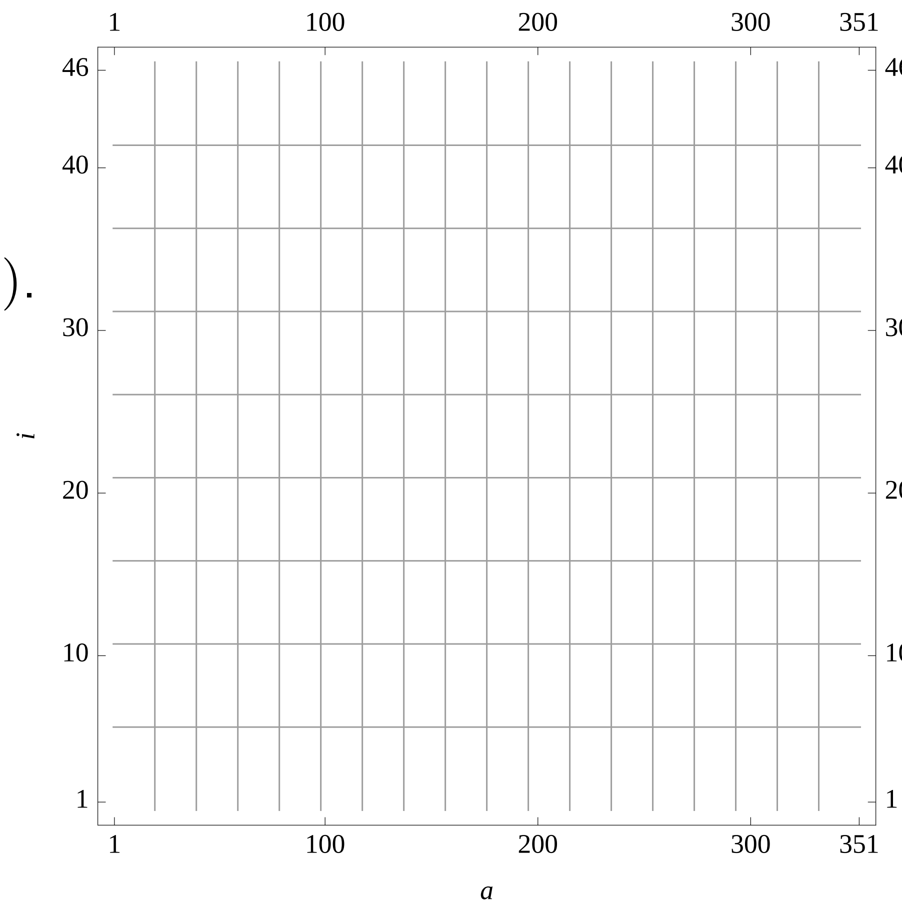
  $1(7100\text{kms}), 351(42000\text{kms})$.

  Each line 2000 kms.
- ☐ Vertical axis: i

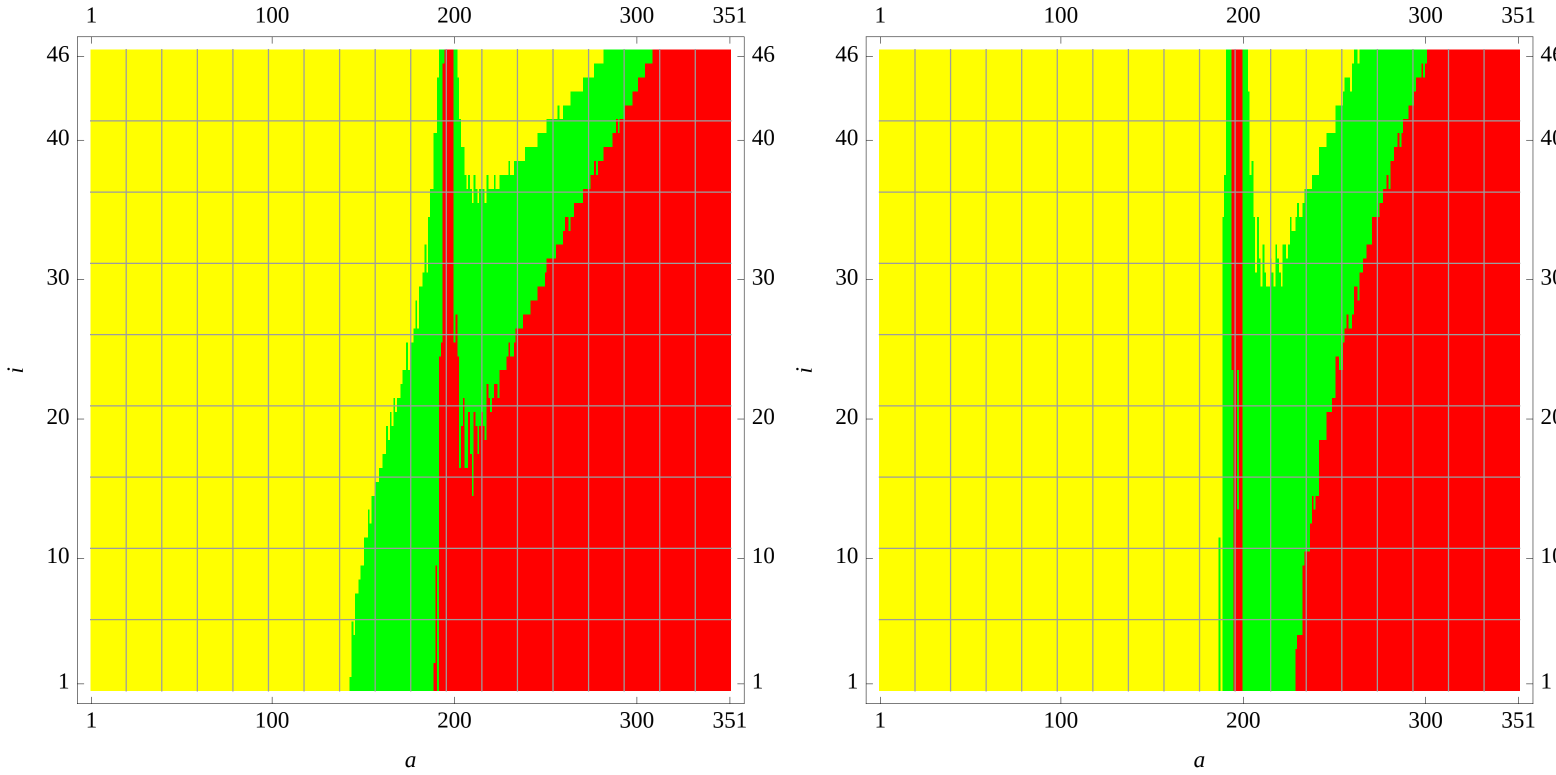
  $1(10^\circ), 46(100^\circ)$.

  Each line 10° .
- ☐ Yellow color: error $<$ 1 km.
- ☐ Green color:

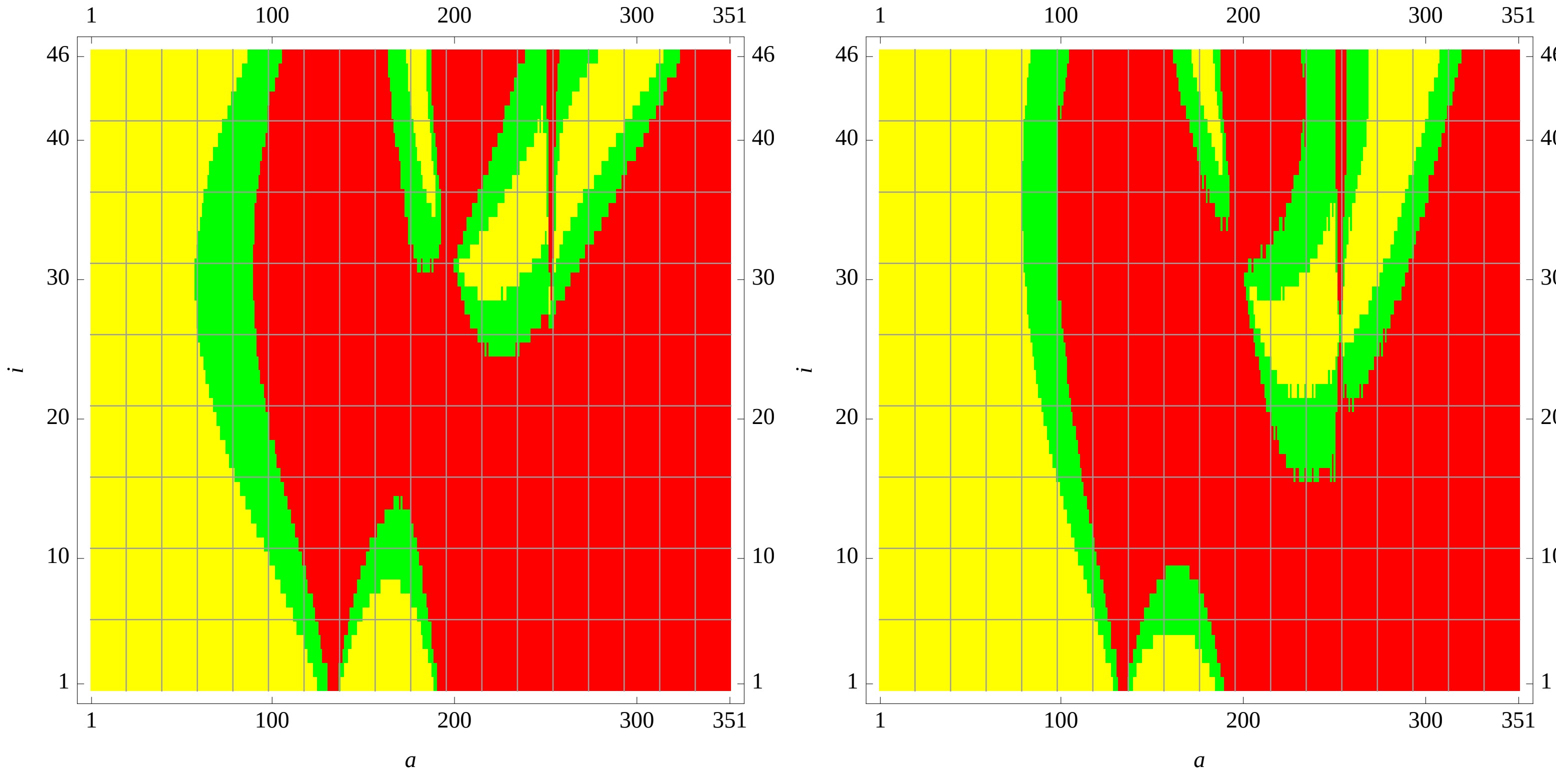
  1 km $\leq$ error $\leq$ 2 km.
- ☐ Red color: error $>$ 2 km.



Tesseral model 2 (2x2) Orders of theory: 6,7



Tesseral model 4 (4x4) Orders of theory: 6,7



Tesseral model 5 (5x5) Order of theory: 6