Development of the Semi-analytical Satellite Theory and Applications

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What is a Semi-Analytical Satellite Theory?

- **Cowell Equations of Motion**

$$x'' = -\frac{\mu}{r^3} x + Q(x, x', t)$$

- **Semi-Analytical Satellite Theory**
  - **Equations of Motion for the Mean Elements**

$$\frac{d\bar{a}_i}{dt} = \sum_{j=1}^{N} \varepsilon^j A_{i,j}(\bar{a}) + O(\varepsilon^{N+1}) \qquad (\mathrm{i} = 1,2,\ldots,5)$$

  - **Analytical Expressions for the Short-Periodic Motion**

$$a_i = \bar{a}_i + \sum_{j=1}^{N} \varepsilon^j \eta_{i,j}(\bar{a}, \bar{\lambda}) + O(\varepsilon^{N+1}) \qquad (\mathrm{i} = 1,2,\ldots 5)$$

What is a Semi-Analytical Satellite Theory? -- Short-Period Motion Formulas

- **Zonal Harmonics (closed form)**

$$\Delta a_i = C_{i,0} + S_{i,0}(L - \lambda) + \sum_{k=1}^{2N+1} \left[C_{i,k} \cos(kL) + S_{i,k} \sin(kL) \right]$$

- **Tesseral m-Dailies (closed form)**

$$\Delta a_i = \sum_{k=1}^{M} \left[C_{i,k} \cos(k\theta) + S_{i,k} \sin(k\theta) \right]$$

- **Tesseral Linear Combination Terms**

$$\Delta a_i = \sum_{k} \sum_{t} \left[C_{i,t,k} \cos(t\lambda - k\theta) + S_{i,t,k} \sin(t\lambda - k\theta) \right]$$

- **Lunar-Solar Point Masses (closed form)**

$$\Delta a_i = C_{i,0} + \sum_{k=1}^{N+1} \left[C_{i,k} \cos(kF) + S_{i,k} \sin(kF) \right]$$

What is a Semi-Analytical Satellite Theory? – State Transition Matrix

- **Semi-analytical Theory for the Partial Derivatives**

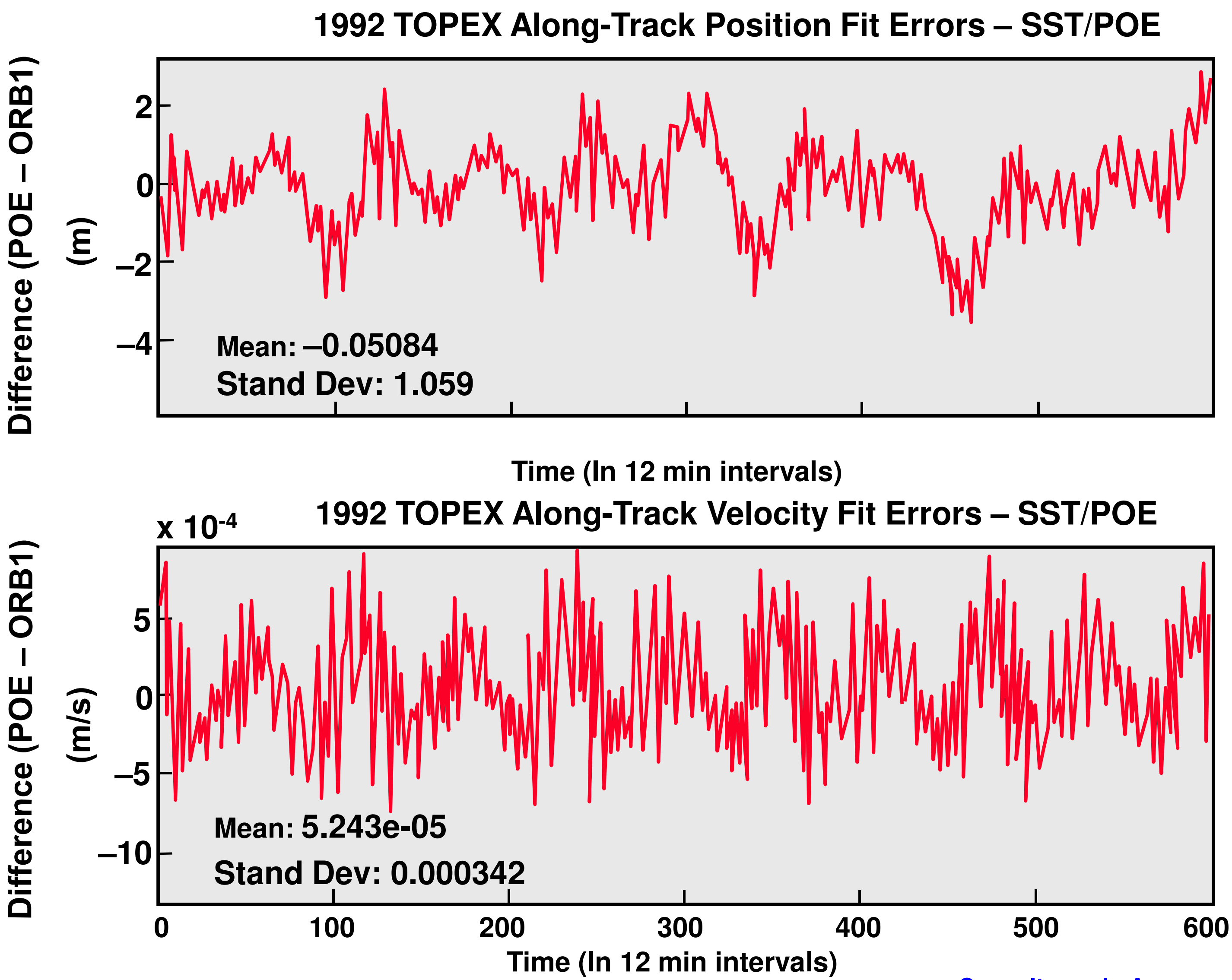
$$\frac{d}{dt}B_2 = AB_2 \text{ with } \left[B_2\right]_{t_0} = I$$

$$\frac{d}{dt}B_3 = AB_3 + \mathrm{D} \quad \text{with } \left[B_3\right]_{t_0} = [0]_{6x(l-6)}$$

What is a Semi-Analytical Satellite Theory? -- Interpolator

- Interpolator Structure
  - Hermite Interpolators for the mean elements
  - Lagrangian Interpolators for the Fourier coefficients in the short-periodic expansions
  - Hermite interpolators for the mean element state transition matrix and the mean element partial derivatives
  - Position and Velocity interpolators for dense output grids
- Interpolator Strategy
  - Construct the first set of interpolators
  - Use until the output request time is outside the current interpolator interval
  - Construct interpolators for the next interpolation interval

Least Squares Fit of SST Theory to TOPEX Orbit – Along-Track Fit Errors



Semi-analytical Orbit Propagator

- **Analytical Averaging:** $$\frac{d\bar{a}_i}{dt} = \sum_{j=1}^{6} (\bar{a}_i, \bar{a}_j) \frac{\partial}{\partial \bar{a}_j} \left[\frac{1}{2\pi} \int_0^{2\pi} R d\bar{M} \right]$$

- **Numerical Averaging:** $$\frac{d\bar{a}_i}{dt} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \bar{a}_i}{\partial x} \bullet Q d\bar{M}$$

Expansion of the Geopotential in Equinoctial Elements

- **Expression in spherical harmonics relative to the ECI frame**
  - radial distance, latitude, and longitude
- **Rotation of the spherical harmonics to the equinoctial orbital frame**
  - Jacobi polynomials replace the Kaula inclination functions
  - stable recursion formulas are available in the mathematical literature
- **Products of radius to a power times sin or cos of true longitude (and multiples) are expanded as a Fourier series in the mean longitude**
  - Modified Hansen coefficients replace the Kaula eccentricity functions
  - stable recursion formulas due to Hansen (1855) were rediscovered in the astronomical literature
  - Jozef Van Der Ha translation of the Hansen manuscript to English (1978)

Rotation of the Spherical Harmonics

- **Generating function**

$$P_{nm}(\sin\phi)e^{jm\alpha} = \sum_{r=-n}^{n} \frac{(n-r)!}{(n-m)!} P_{n,r}(0) S_{2n}^{(m,r)}(p,q) e^{jrL}$$

- **where (for m $\geq$ 0)**

$$S_{2n}^{(m,r)}(p,q) = (1+p^2+q^2)^r (p-jq)^{m-r} P_{n+r}^{(m-r,-m-r)}(\gamma)$$

r $\leq$ -m

$$S_{2n}^{(m,r)}(p,q) = \frac{(n+m)!(n-m)!}{(n+r)!(n-r)!}(1+p^2+q^2)^{-m}(p-jq)^{m-r} P_{n-m}^{(m-r,r+m)}(\gamma)$$

-m $\leq$ r $\leq$ m

$$S_{2n}^{(m,r)}(p,q) = (-1)^{m-r}(1+p^2+q^2)^{-r}(p+jq)^{r-m} P_{n-r}^{(r-m,r+m)}(\gamma)$$

r $\geq$ m

- **and** $\gamma = \cos i$

Hansen Coefficients

- **Generating Function**

$$(r/a)^n e^{isf} = \sum_{t=-\infty}^{t=+\infty} X_t^{n,s} e^{itM}$$

- **Recursion Relation employed in Tesseral Resonance**

$$n[(n-2)^2 - m^2]\cos^2\varphi X_t^{n-4,m} = n(n-2)(2n-3)X_t^{n-3,m}$$
$$-(n-1)[n(n-2) + 2tm\cos\varphi]X_t^{n-2,m} + t^2(n-2)X_t^{n,m}$$

- **Special Case (t=0; zonal harmonics)**

$$n[(n-2)^2 - m^2]\cos^2\varphi X_0^{n-4,m} = n(n-2)(2n-3)X_0^{n-3,m}$$
$$-(n-1)[n(n-2)]X_0^{n-2,m}$$

Modified Newcomb Operator Expansion

- **Standard Expansion**

$$X_t^{n,s} = e^{|t-s|} \sum_{i=0}^{\infty} X_{i+a,i+b}^{n,s} e^{2i}$$

$$a = \frac{|t-s| + t - s}{2}$$

$$b = \frac{|t-s| - (t-s)}{2}$$

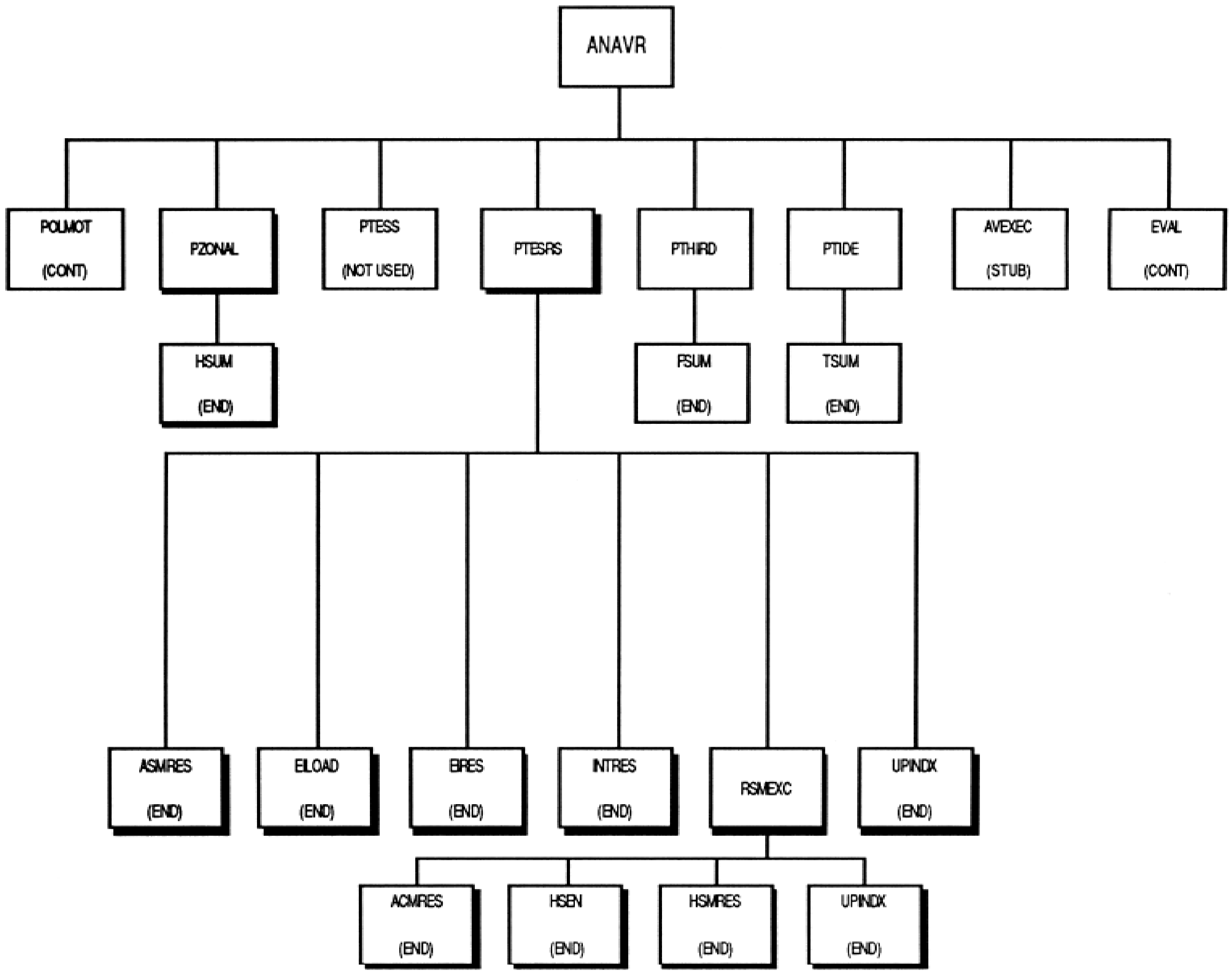
- **New Expansion**

$$X_t^{n,s} = (1-e^2)^{n+3/2} e^{|t-s|} \sum_{i=0}^{\infty} Y_{i+a,i+b}^{n,s} e^{2i}$$

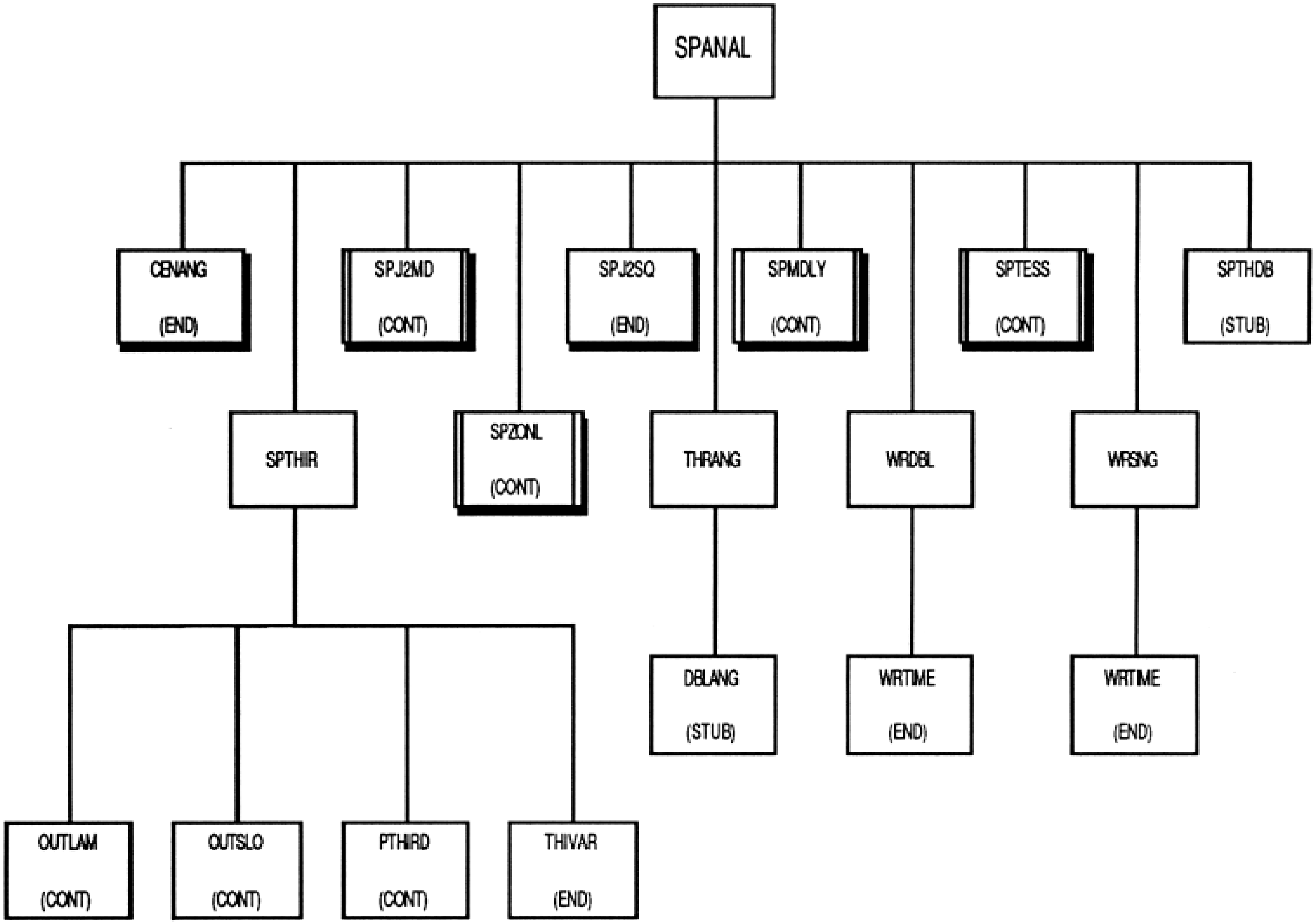
Convergence for the set of Hansen coefficients $X_t^{n,s}$ at e = 0.7 with relative accuracy 1.0D-5

n	s	t	N	Y
-3	0	1	17	6
-3	2	1	9	9
-4	-1	1	20	7
-4	1	1	20	7
-5	0	2	22	6
-5	2	2	23	7
-6	-1	1	25	6
-6	1	1	26	5
-6	-1	2	25	7
-6	1	2	25	7

Semi-analytical Theory -- Analytical Averaging Software Architecture



Semi-analytical Theory -- Analytical Short-Periodic Model Software Architecture



SPZONL Inputs and Outputs

- Subroutine SPZONL(COSC, SINC, NMAX, LMAX, KMAX)
- COSC is the output array of cosine coefficients in the true longitude expansion
- SINC is the output array of sine coefficients in the true longitude expansion
- NMAX is the input maximum power of (R_e/a)
- LMAX is the input maximum power of the eccentricity
- KMAX is the input maximum power of exp (iL)
- The slowly varying portion of the mean equinoctial element set

SPZONL Example Output

EXPANSION IN SINES AND COSINES OF J*L

TIME AFTER EPOCH = 0 DAYS 0 HOURS 0 MINUTES 0.00 SECONDS. ELAPSED TIME = 0.00 SECONDS.

NON-TRIGONOMETRIC COEFFICIENTS

TERM	DA (KM)	DH	DK	DP	DQ	DLAMBDA (RAD)
CONSTANT	-1.7936800143E-05	-4.5316345886E-07	1.0244989879E-06	-8.4134274777E-12	-2.4872353647E-12	-1.6180058541E-11
L - LAMBDA	0.000000000	7.6236983515E-07	2.9112772927E-07	-2.8571165071E-05	6.0355983045E-05	-7.9599431963E-04

COSINE COEFFICIENTS

J	DA (KM)	DH	DK	DP	DQ	DLAMBDA (RAD)
1	2.2488807229E-02	2.8332334139E-04	-9.6149267404E-04	4.4577677456E-08	-2.4719456990E-08	9.0754448003E-07
2	-6.229395462	5.4751551215E-07	1.7814934626E-06	-3.0177991523E-05	1.4285582535E-05	-8.7576508376E-04
3	4.7793807529E-03	-6.6108842065E-04	-5.4179555325E-04	7.8927955274E-09	-8.2398189967E-09	7.8275146435E-07
4	3.3313343668E-07	5.3441939672E-07	1.7814962564E-07	0.000000000	0.000000000	-2.8850057891E-10
5	-5.2497420954E-10	-9.6166834368E-11	4.1391427535E-12	0.000000000	0.000000000	4.5295356765E-14

SINE COEFFICIENTS

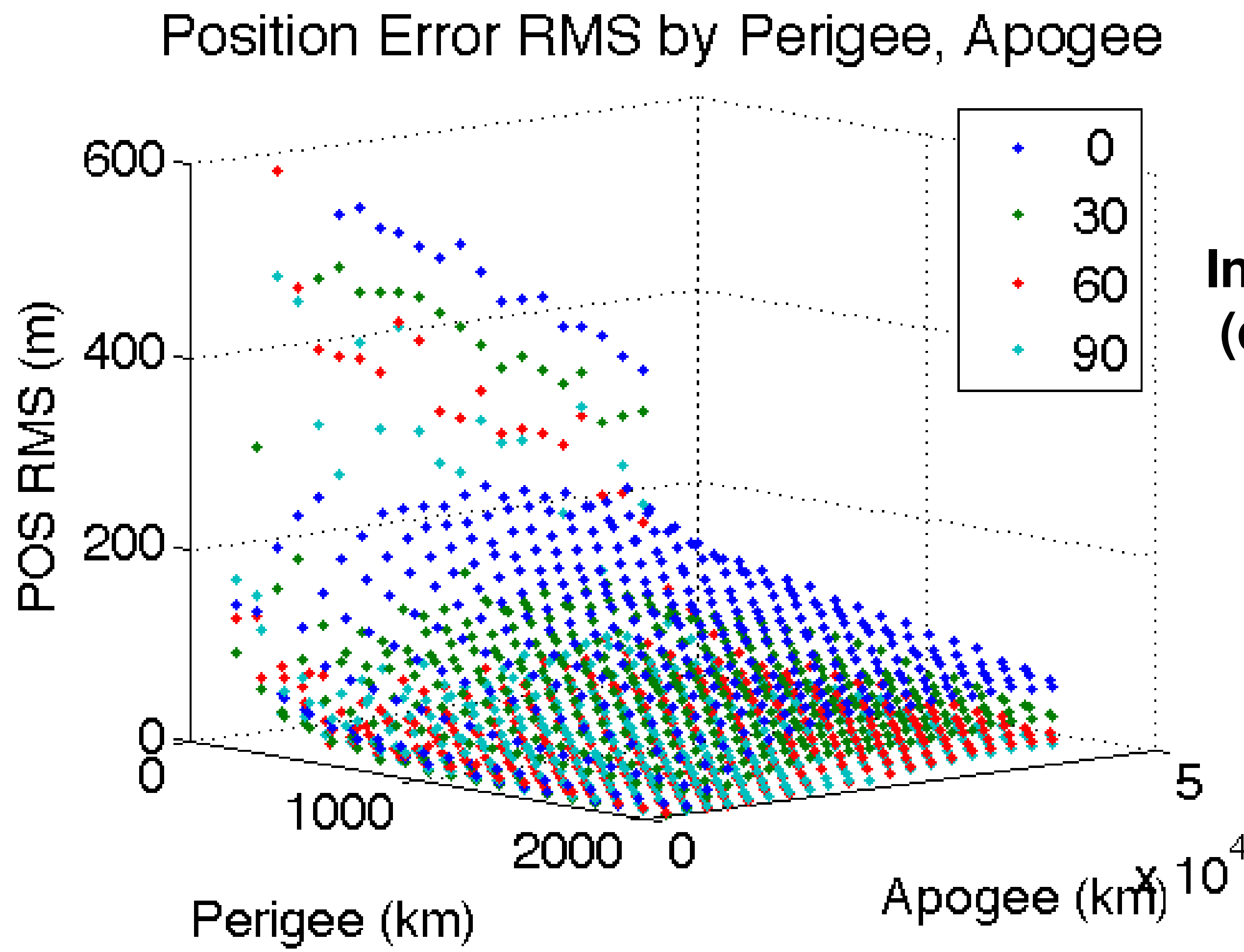
J	DA (KM)	DH	DK	DP	DQ	DLAMBDA (RAD)
1	-1.1080264328E-02	-4.9709688639E-04	2.8332358407E-04	2.4719456990E-08	-9.1934450620E-08	1.9563249626E-06
2	7.600987375	1.5547734378E-06	-1.2276380687E-06	1.4285582535E-05	3.0177991523E-05	-7.1773391564E-04
3	-1.4337350939E-02	-5.4179556103E-04	6.6108860131E-04	-8.2398189967E-09	-7.8927955274E-09	2.6093155308E-07
4	7.7398606271E-06	1.7814962564E-07	-5.3441939672E-07	0.000000000	0.000000000	1.2417431523E-11
5	-1.2151786653E-09	4.1391427535E-12	9.6166834368E-11	0.000000000	0.000000000	-1.9568228766E-14

Short-Periodic Coefficient Interpolator Errors

Points	Step (days)	Zonal Errors*		
		ΔC_a (m)	ΔC_p (m)	ΔC_λ (m)
2	.25	.86	.064	.76
	.5	3.5	.26	3.1
	1.	14.0	1.0	12.
3	.25	.012	0	.011
	.5	.098	.004	.036
	1.	.78	.029	.69
4	.25	0	0	0
	.5	.002	0	.002
	1.	.015	.004	.021

***These errors are the position errors caused by errors in the interpolated coefficients For the 2 lambda short-periodics. The stepsize is the interval between successive interpolation points.**

Truncated J_2^2 Modeling in DSST Leads to Errors for Eccentric Orbits



Inclination (degrees)

J2-squared closed form development

- **J2-squared contributions to the averaged equations of motion**

$$A_{i,2} = \left\langle \sum_{k=1}^{6} \eta_{k,1} \frac{\partial F_i}{\partial \bar{a}_k} \right\rangle_{\bar{\lambda}}$$

$$A_{6,2} = \left\langle \sum_{k=1}^{6} \eta_{k,1} \frac{\partial F_6}{\partial \bar{a}_k} + \frac{15}{4} \frac{\bar{n}}{\bar{a}_1^2} \eta_{1,1}^2 \right\rangle_{\bar{\lambda}}$$

- **The F_i's are the RHS of the osculation equinoctial element equations of motion**

Outline for J_2^2 Modeling Enhancement for DSST

- **Develop closed form expressions for J2 disturbance potential functions using Zeis' *maxima* blocks**
  - **Fischer (1998), updated in 2011**
- **Develop closed-form solutions to VOP equations of motion (F functions) using Zeis' maxima blocks**
  - **Fischer (1998), updated in 2011**
- **Determine partials of F functions with respect to orbital elements**
  - **analytical approach**
  - **finite difference approach for testing**
- **Determine the first order zonal periodic function C and S coefficients using Slutsky's SPZONL routine**
- **Develop summations of the products of the first order zonal periodic functions with the partials of the F functions**
- **Average the summations over the fast variable**

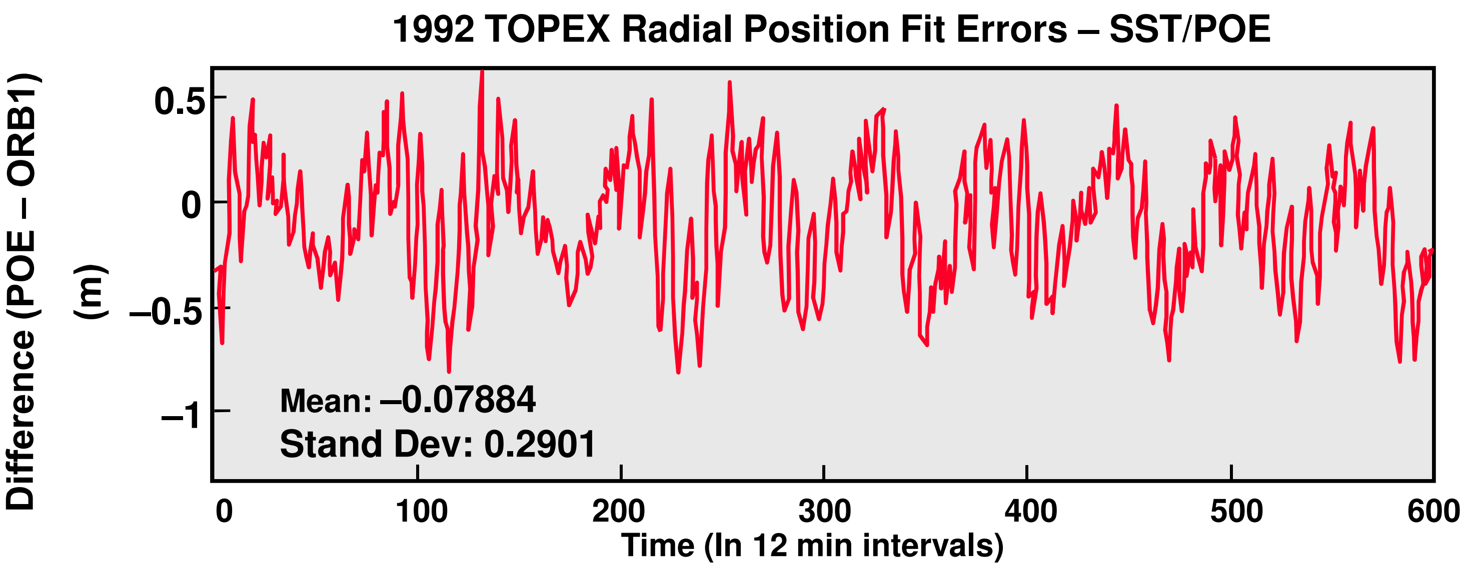
TOPEX External Reference Orbits

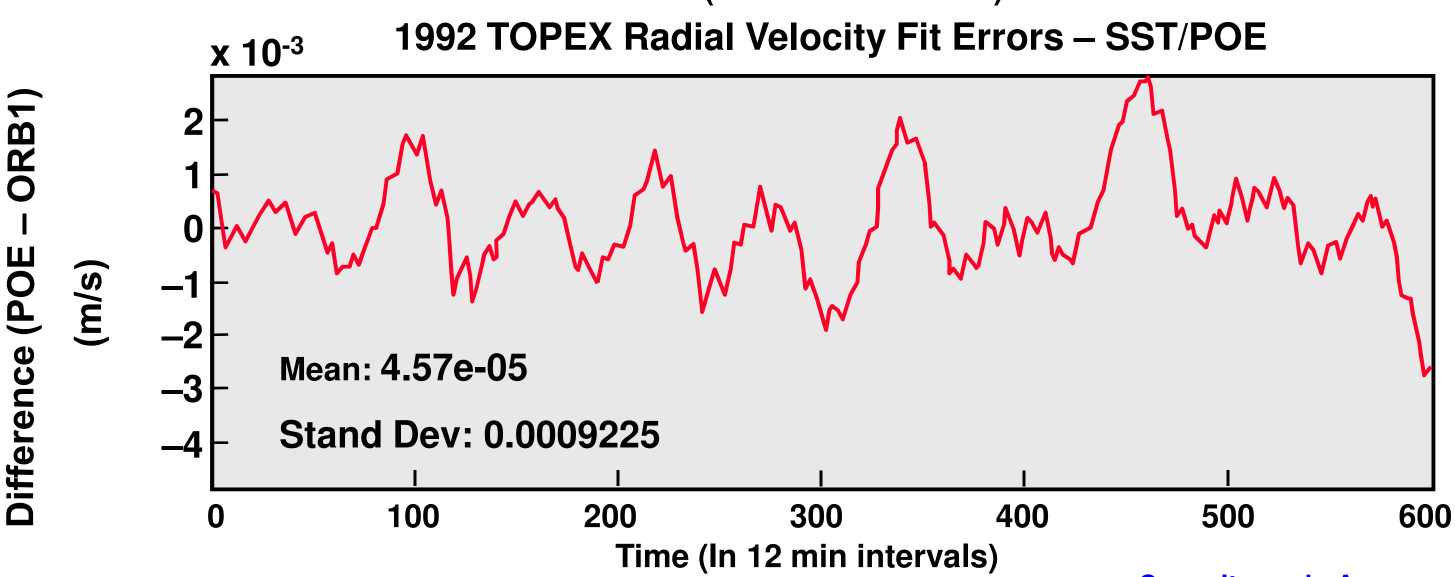
- Intensive tracking
  - Satellite laser ranging
  - Differential GPS
  - DORIS on-board device (TOPEX)
- Very precise force models
  - JGM-2 70x70 Gravity Field
  - Lunar-Solar Point Masses
  - Solid Earth Tides
  - Atmosphere Drag (Jacchia-Roberts density)
  - Solar Radiation Pressure (Conical Model)
  - Earth Radiation Pressure
  - Ocean Tides
  - Rotational Deformation
- 15 cm maximum error, over the whole orbit (TOPEX)
  - NASA Goddard, University of Texas, CNES

Semianalytical Satellite Theory Modeling of the TOPEX Orbit

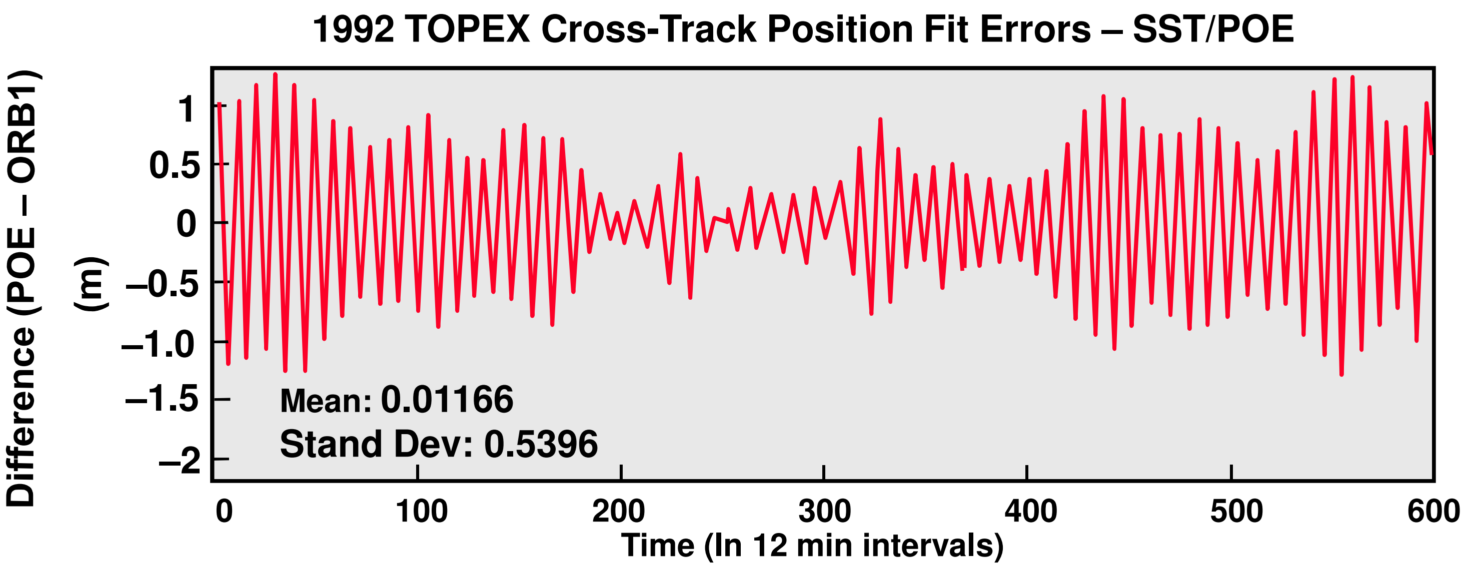
- Dynamics
  - Mean Element Equations of Motion
    - 50 x 50 Geopotential (truncated JGM 2)
    - Lunar-Solar Point Masses
    - Atmosphere Drag (Jacchia-Roberts)
    - Solar Radiation Pressure
    - Solid Earth Tides (solar & lunar terms)
  - Short Periodic Terms
    - Zonals, Tesseral M-dailies, Tesseral Linear Combinations
    - J2 / Tesseral M-daily Coupling
  - Integration Coordinate System – Mean of J2000.0
- Solve-for Vector
  - Mean Equinoctial Elements
  - Solar Radiation Pressure Coefficient
  - Drag Coefficient

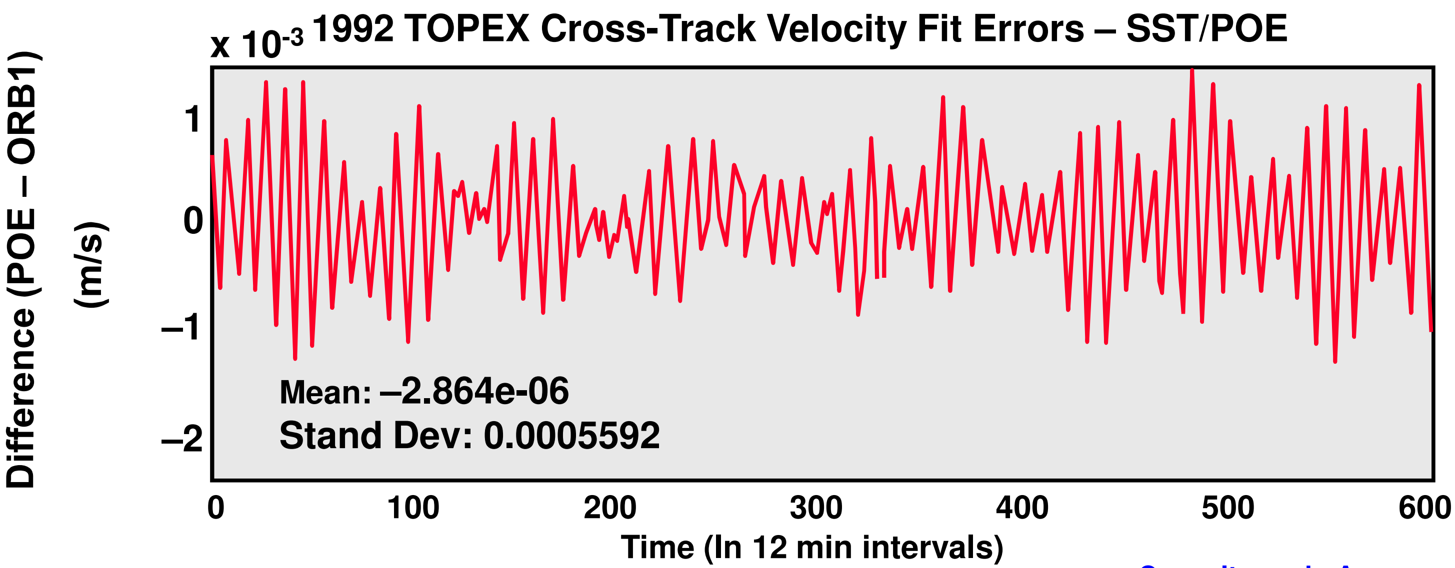
Least Squares Fit of SST Theory to TOPEX External Reference Orbit – Radial Fit Errors



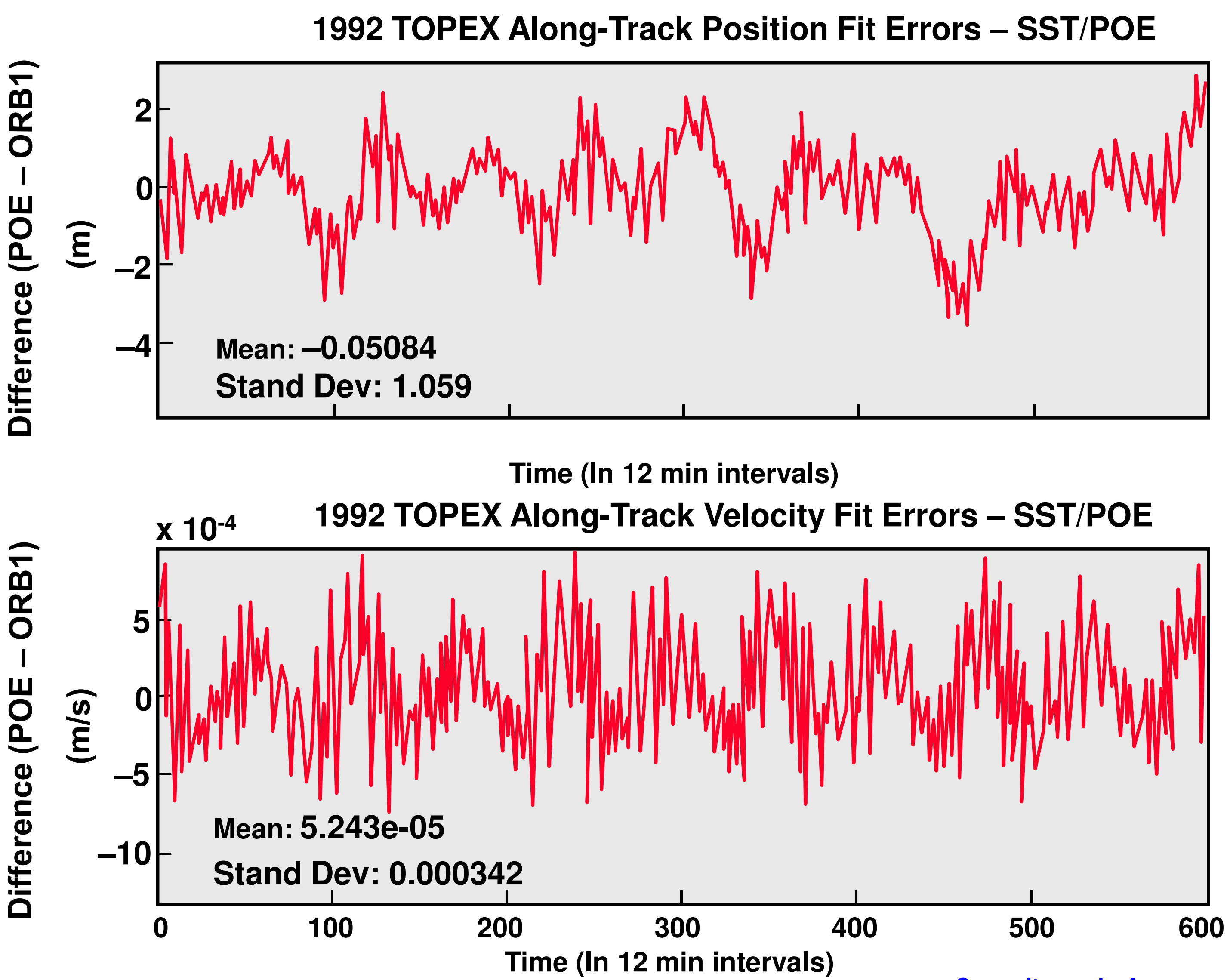


Least Squares Fit of SST Theory to TOPEX Orbit – Cross-Track Fit Errors

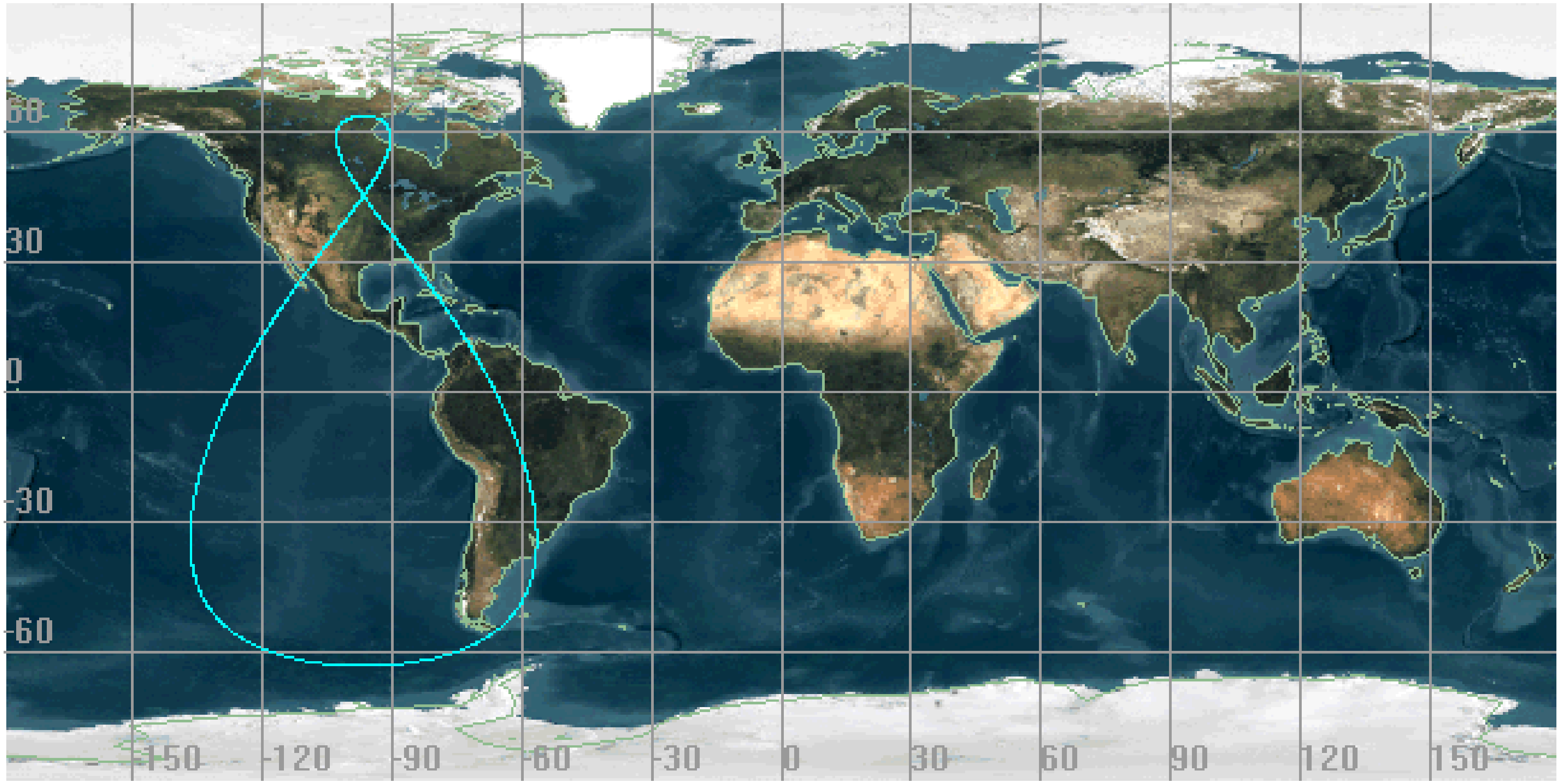




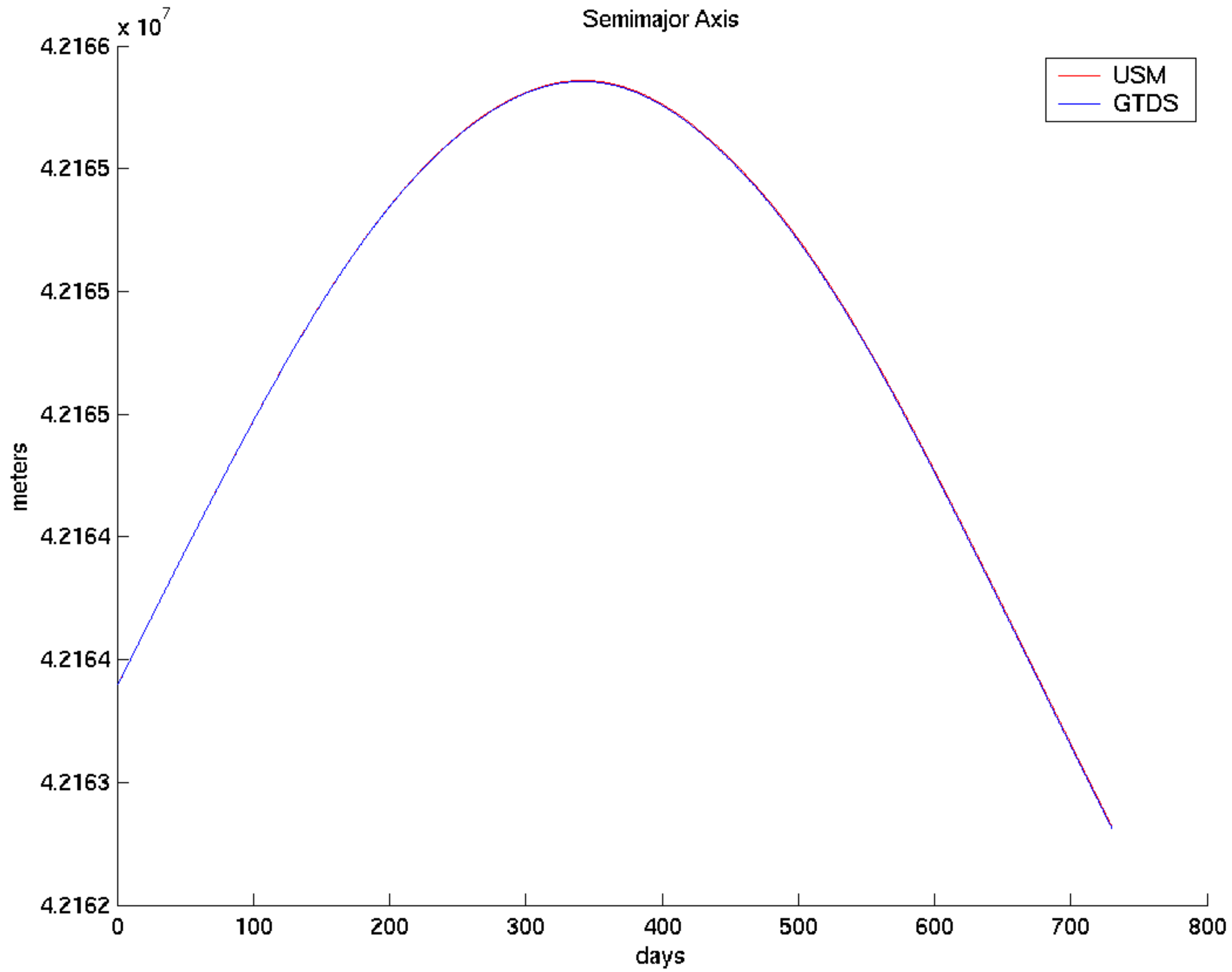
Least Squares Fit of SST Theory to TOPEX Orbit – Along-Track Fit Errors



Sirius 24 hr Elliptical Orbit Ground Track



DSST and USM Mean Semi-Major Axis Time Histories (Sirius Case)

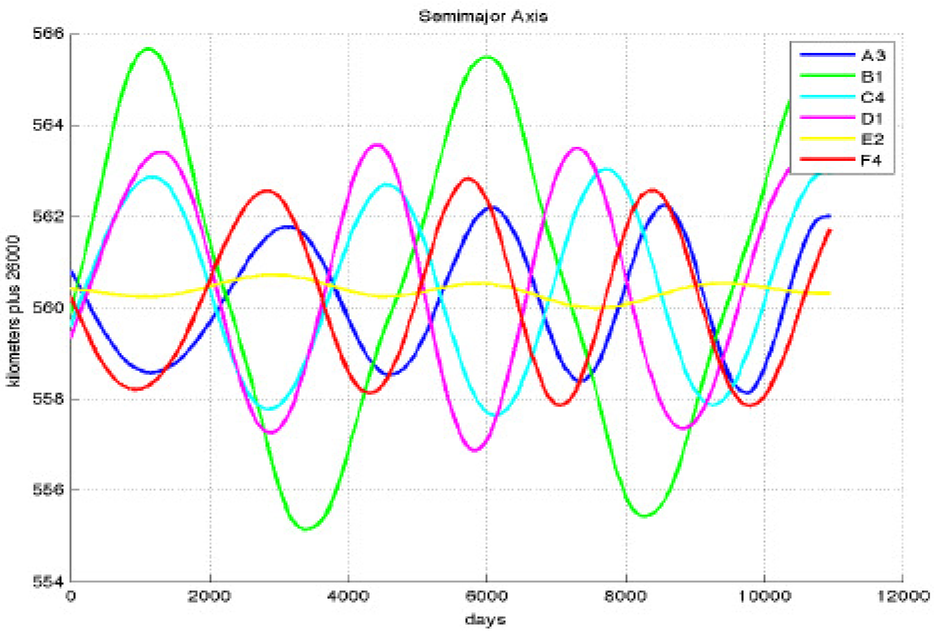


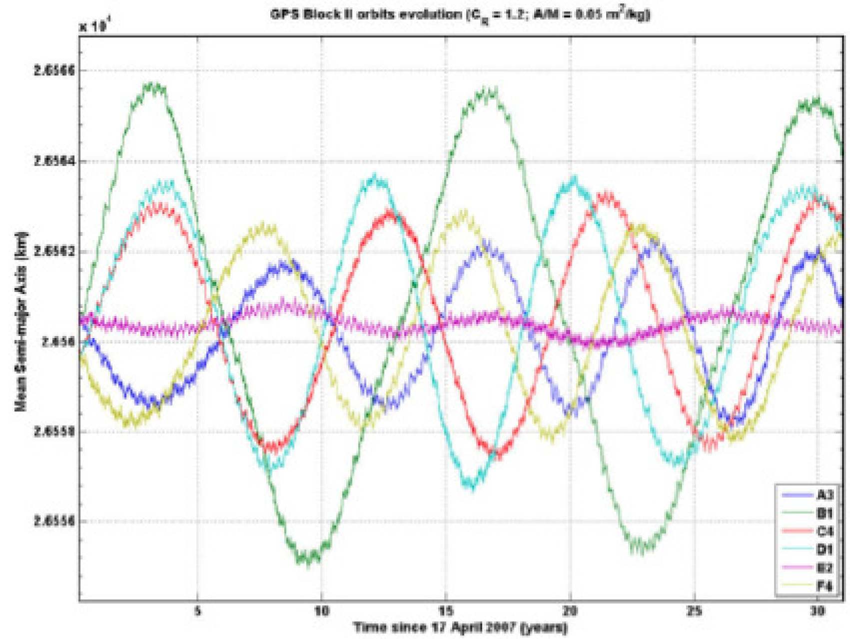
Preliminary DSST Calibration for GPS orbits

- Compare with Anselmo & Pardini 2009 perturbation model:
  - 16×16 truncation of the EGM-96 gravity model
  - lunar and solar third body attraction, SRP with eclipses
  - area-to-mass ratios set to 0.05 m^2/kg
- Significant sample of GPS orbits: one for each Block II plane
  - appropriate spread in a, λ, i
- Epoch referred to April 16, 2007; $\Delta a = a - 26559$ km

	UTC (h:m:s)	Δa	$10^3\, e$	i (deg)	ω (deg)	Ω (deg)	M (deg)
A3	27:05:55.335	1.826	9.6227	56.0150	159.9943	72.5607	200.3662
B1	00:46:42.433	0.890	4.1632	55.2000	315.4985	130.7562	44.2377
C4	07:29:14.720	0.616	2.6456	54.9919	189.3242	190.2977	170.6826
D1	35:03:07.094	0.355	8.7123	54.2489	130.4077	249.6308	230.3125
E2	17:28:58.794	1.432	4.8672	54.5093	265.1898	312.7360	94.2809
F4	23:06:57.974	1.259	4.6480	55.6068	150.7686	9.9670	230.3125

GPS Mean Semi-major Axes, 30 year Evolution





Left: DSST runs; Right: SATRAP runs from Anselmo & Pardini 2009

Conclusions

- We have described a comprehensive semi-analytical satellite theory that uses the mean equinoctial variables as the fundamental constants
- Extensive force modeling – 50 x 50 geopotential, third-body point masses, solid Earth tides, drag, solar radiation pressure
- The theory has been tested against a variety of data – most recently SLR and precise GPS solutions
- The theory has proven to be portable to a variety of computer platforms, operating systems, and organizations – currently employed under Linux
- The theory is the basis of the Aerospace Corp MEANPROP program
- The mean equinoctial elements have proven advantageous in Orbit Determination applications – Batch Least Squares, Extended Semi-analytical Kalman Filters (ESKF), BSESKF

Conclusions -- Limitations

- Accessibility – currently thru the Linux GTDS R&D or the DSST Standalone programs
- Current J_2-squared eccentricity truncation
- Fortran 77 source code limits accessibility to modern computer hardware concepts such as Graphical Processor Unit (GPU)

Future Work

- Partially address accessibility through Web access to the DSST Standalone (Astrodyn Web framework)
  - Presentation next week by J.F. San Juan (U. Rioja) at the IAF IAC in Capetown
- Evolve the current source code to more modern Fortran (90, 95, or 2003)
- Develop an Object Oriented DSST code – C++, JAVA
- Develop an Open Source DSST capability
- Review early work on parallel algorithms to take advantage of the new computer hardware
- J2-squared development
- Evolve the state transition matrix capabilities
- Arbitrary central body capability

Acknowledgement

- *Wayne McClain* – application of the Generalized Method of Averaging in DSST
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- *Dr. Ron Proulx* – recursive tesseral resonance, tesseral m-daily and linear combination short-periodic models, Hansen coefficients
- *Stephen Taylor* – integration of semi-analytical satellite theory and Extended Kalman Filter concepts
- *Dan Fonte* – expansion of the recursive models to 50 x 50
- *Scott Carter* – integration of 50 x 50 and J2000 integration coordinate system
- *Jack Fischer* – highly elliptical orbits
- *Zachary Folcik* – DSST in the Linux environment, testing of DSST vs very high accuracy observation data at MIT LL, BSESKF development