Analytical Integration of the Osculating Lagrange Planetary Equations for the Third Body and the SRP Perturbations in the Orbital Motion

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Introduction

story

he field of orbital motion perturbation methods, a half century of works has produced a lo analytical theories. These theories are either based on Hamiltonian developments and ries expansions of the perturbing functions (Von-Zeipel, Brouwer, Lie-Deprit), or use rative approximation algorithms (Kozaï, Kaula). The differential system of the Lagrange anetary Equations has also been solved using Cook's algorithms.

ctual

nerally speaking, analytical theories have difficulties to deal with high eccentricity orbits, e to series expansions or due to the large number of terms in closed forms.

lving the high eccentricity problem

e present work com back to the original Lagrange's equations. The Restricted Three Boo oblem for an high eccentricity satellite orbit is solved in osculating elements.

ar, lunar and direct Solar Radiation Pressure potentials are simplified.

Lagrange Planetary Equations in the Restricted Three Body Problem

ystem of osculating equations

$$
\begin{pmatrix} \dfrac{da}{dt} \\ \dfrac{de}{dt} \\ \dfrac{di}{dt} \\ \dfrac{d\Omega}{dt} \\ \dfrac{d\omega}{dt} \\ \dfrac{dM}{dt}-n \end{pmatrix}
=
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \dfrac{-2}{na} \\
0 & 0 & 0 & 0 & \dfrac{\sqrt{1-e^2}}{na^2 e} & \dfrac{e^2-1}{na^2 e} \\
0 & 0 & 0 & \dfrac{1}{na^2\sqrt{1-e^2}\sin i} & \dfrac{-\cot i}{na^2\sqrt{1-e^2}} & 0 \\
0 & 0 & \dfrac{-1}{na^2\sqrt{1-e^2}\sin i} & 0 & 0 & 0 \\
0 & -\dfrac{\sqrt{1-e^2}}{na^2 e} & \dfrac{\cot i}{na^2\sqrt{1-e^2}} & 0 & 0 & 0 \\
\dfrac{2}{na} & \dfrac{1-e^2}{na^2 e} & 0 & 0 & 0 & 0
\end{pmatrix}
\times
\begin{pmatrix} \dfrac{\partial R}{\partial a} \\ \dfrac{\partial R}{\partial e} \\ \dfrac{\partial R}{\partial i} \\ \dfrac{\partial R}{\partial \Omega} \\ \dfrac{\partial R}{\partial \omega} \\ \dfrac{\partial R}{\partial M} \end{pmatrix}
$$

e second members of the equations are the partial derivatives of the third body potential
e direct Solar Radiation Pressure potential in a quasi-inertial frame.

uasi-inertial frame

s is the Veis quasi_inertial frame fixed at the epoch 1950 00h00’00’’.

phemerides of the Sun and the Moon

sitions of the Sun and the Moon are from Newcomb and Brown theories.

Potentials (1)

Simplified potential of the Sun and the Moon

$$U_s = \frac{\mu_s}{r_s}[1 + \frac{3}{2}(\frac{r}{r_s})^2(\cos^2\theta - \frac{1}{3})]$$

ne inertial frame :

$$= \frac{\mu_s}{r_s}(1 + \frac{3}{2}(\frac{\frac{a(1-e^2)}{1+e\,cos(v)}}{r_s})^2((\frac{1}{r_s})^2$$

$$((cos(\omega + v)cos(\Omega) - sin(\omega + v)cos(i)sin(\Omega))x_s + (cos(\omega + v)sin(\Omega) + sin(\omega + v)cos(i)cos(\Omega))y_s + sin(\omega + v)sin(i)z$$

$$-\frac{1}{3}))$$

ample of a partial derivative : $\frac{\partial U_s}{\partial a}$

s*a/rs^5*(1-e^2)^2*(xs^2*(-cos(i)*sin(Ω)*sin(v+ω))^2+2*ys^2*cos(Ω)*cos(i)*cos(v+ω)*sin(Ω)*sin(v+ω)-2*xs^2*cos(
i)*cos(v+ω)*sin(Ω)*sin(v+ω)+2*xs*ys*cos(Ω)^2*cos(i)*cos(v+ω)*sin(v+ω)-2*xs*ys*cos(i)*cos(v+ω)*sin(Ω)^2*sin(v
2*xs*ys*cos(Ω)*cos(i)^2*sin(Ω)*sin(v+ω)^2+2*xs*ys*cos(Ω)*cos(v+ω)^2*sin(Ω)+ys^2*cos(Ω)^2*cos(i)^2*sin(v+ω)
s^2*cos(v+ω)^2*sin(Ω)^2+xs^2*cos(Ω)^2*cos(v+ω)^2+2*ys*zs*cos(Ω)*cos(i)*sin(i)*sin(v+ω)^2-2*xs*zs*cos(i)*sin(
)*sin(v+ω)^2+2*ys*zs*cos(v+ω)*sin(Ω)*sin(i)*sin(v+ω)+2*xs*zs*cos(Ω)*cos(v+ω)*sin(i)*sin(v+ω)+zs^2*sin(i)^2*sin(
^2+1*(-(rs^2))/3)/(1+e*cos(v))^2

Potentials (2)

Simplified potential of the direct Solar Radiation Pressure

$$U_{SRP} = -\sigma \frac{A}{M} r \cos\theta$$

he inertial frame :

$$= -\sigma \frac{A}{M} \frac{a(1-e^2)}{1+e\ cos(v)} (\frac{1}{r_s})$$

$$((cos(\omega+v)cos(\Omega) - sin(\omega+v)cos(i)sin(\Omega))x_s + (cos(\omega+v)sin(\Omega) + sin(\omega+v)cos(i)cos(\Omega))y_s + sin(\omega+v)sin(i)z_s$$

ample of a partial derivative : $\frac{\partial U_{SRP}}{\partial a}$

)*A/M*(1-e^2)/rs*(ys/(1+e*cos(v))*cos(Ω)*cos(i)*sin(v+ω)-xs/(1+e*cos(v))*cos(i)*sin(Ω)*sin(v+ω)+zs/(1+e*cos(v))*sin(
+ω)+ys/(1+e*cos(v))*cos(v+ω)*sin(Ω)+xs/(1+e*cos(v))*cos(Ω)*cos(v+ω))

Integration of the Osculating Lagrange Planetary Equations (1)

riable of integration : the true anomaly

h

$$dt = \frac{1}{n}\,\frac{dM}{dv}\,dv \qquad \frac{dM}{dv} = \frac{(1-e^2)^{\left(\frac{3}{2}\right)}}{(1+e\ \cos(v))^2}$$

egrals

mple for $\frac{1}{n}\int \frac{\partial U_s}{\partial a}\,\frac{dM}{dv}\,dv$ **where the characteristical primitives are :**

$$\int \frac{1}{\ldots\cos(x))^4}\,dx =$$

$$\ldots n(x)\left(8\,b^4 + \left(4\,b^2+11\right) b^2 \cos(2\,x) + \right.$$

$$6\left(b^2+9\right) b \cos(x) +$$
$$b^2+36)\big) / \left(2\,(b-1)^3\right.$$
$$\left.(b+1)^3\,(b\cos(x)+1)^3\right) -$$
$$\frac{6\left(3\,b^2+2\right)\tanh^{-1}\left(\frac{(b-1)\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-1}}\right)}{\left(b^2-1\right)^{7/2}}\Bigg) +$$

ant

$$\int \frac{\cos^2(\omega+x)}{(1+b\cos(x))^4}\,dx =$$

$$\big(24\,b^7\sin(x-2\,\omega)+8\,b^7\sin(3\,x-2\,\omega) - 24\,b^7\sin(x+2\,\omega) + 8\,b^7\sin(3\,x+2\,\omega)+8\,b^7\sin(3\,x) + 18\,b^6\sin(2\,(x-\omega)) + 18\,b^6\sin(2\,(x+\omega)) - 16\,b^6\sin(2\,\omega) - 27\,b^5\sin(x-2\,\omega) + 9\,b^5\sin(3\,x-2\,\omega) + 117\,b^5\sin(x+2\,\omega) + 9\,b^5\sin(3\,x+2\,\omega)+22\,b^5\sin(3\,x)+54\,b^4\sin(2\,(x-\omega)) + 54\,b^4\sin(2\,(x+\omega))+48\,b^4\sin(2\,\omega)+102\,b^3\sin(x-2\,\omega) - 2\,b^3\sin(3\,x-2\,\omega) - 42\,b^3\sin(x+2\,\omega) - 2\,b^3\sin(3\,x+2\,\omega) - 12\,b^2\sin(2\,(x-\omega)) - 12\,b^2\sin(2\,(x+\omega)) - 48\,b^2\sin(2\,\omega) + 12\left(b^2+9\right)b^4\sin(2\,x) + 6\left(4\,b^4-3\,b^2+24\right)b^3\sin(x) - 24\,b\sin(x-2\,\omega) + 24\,b\sin(x+2\,\omega)+16\sin(2\,\omega)\big) / \left(96\,b^2\left(b^2-1\right)^3(b\cos(x)+1)^3\right) -$$

$$\Big(\left(5\,b^2\cos(2\,\omega)+3\,b^2+2\right) \ldots$$

$$\int \frac{\sin(\omega+x)\cos(\omega+x)}{(1+b\cos(x))^4}\,dx =$$

$$\big(24\,b^7\cos(x+2\,\omega) - 8\,b^7\cos(3\,x+2\,\omega) + 18\,b^6\cos(2\,(x-\omega)) - 18\,b^6\cos(2\,(x+\omega)) + 16\,b^6\cos(2\,\omega) - 117\,b^5\cos(x+2\,\omega) - 9\,b^5\cos(3\,x+2\,\omega) + 54\,b^4\cos(2\,(x-\omega)) - 54\,b^4\cos(2\,(x+\omega)) - 48\,b^4\cos(2\,\omega) + 42\,b^3\cos(x+2\,\omega) + 2\,b^3\cos(3\,x+2\,\omega) - 12\,b^2\cos(2\,(x-\omega)) + 12\,b^2\cos(2\,(x+\omega)) + 48\,b^2\cos(2\,\omega) + 3\left(8\,b^6-9\,b^4+34\,b^2-8\right)b\cos(x-2\,\omega) + \left(8\,b^4+9\,b^2-2\right)b^3\cos(3\,x-2\,\omega) - 24\,b\cos(x+2\,\omega) - 16\cos(2\,\omega)\big) / \left(96\,b^2\left(b^2-1\right)^3(b\cos(x)+1)^3\right) -$$

$$\frac{5\,b^2\sin(\omega)\cos(\omega)\tanh^{-1}\left(\frac{(b-1)\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-1}}\right)}{\left(b^2-1\right)^{7/2}} +$$

constant

$$\int \frac{\sin^2(\omega+x)}{(1+b\cos(x))^4}\,dx =$$

$$\frac{\left(5\,b^2\cos(2\,\omega)-3\,b^2-2\right)\tanh^{-1}\left(\frac{(b-1)\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-1}}\right)}{2\left(b^2-1\right)^{7/2}}$$

$$+\big(-24\,b^7\sin(x-2\,\omega) - 8\,b^7\sin(3\,x-2\,\omega) + 24\,b^7\sin(x+2\,\omega) - 8\,b^7\sin(3\,x+2\,\omega) + 8\,b^7\sin(3\,x) - 18\,b^6\sin(2\,(x-\omega)) - 18\,b^6\sin(2\,(x+\omega)) + 16\,b^6\sin(2\,\omega) + 27\,b^5\sin(x-2\,\omega) - 9\,b^5\sin(3\,x-2\,\omega) - 117\,b^5\sin(x+2\,\omega) - 9\,b^5\sin(3\,x+2\,\omega) + 22\,b^5\sin(3\,x) - 54\,b^4\sin(2\,(x-\omega)) - 54\,b^4\sin(2\,(x+\omega)) - 48\,b^4\sin(2\,\omega) - 102\,b^3\sin(x-2\,\omega) + 2\,b^3\sin(3\,x-2\,\omega) + 42\,b^3\sin(x+2\,\omega) + 2\,b^3\sin(3\,x+2\,\omega) + 12\,b^2\sin(2\,(x-\omega)) + 12\,b^2\sin(2\,(x+\omega)) + 48\,b^2\sin(2\,\omega) + 12\left(b^2+9\right)b^4\sin(2\,x) + \ldots$$

Integration of the Osculating Lagrange Planetary Equations (2)

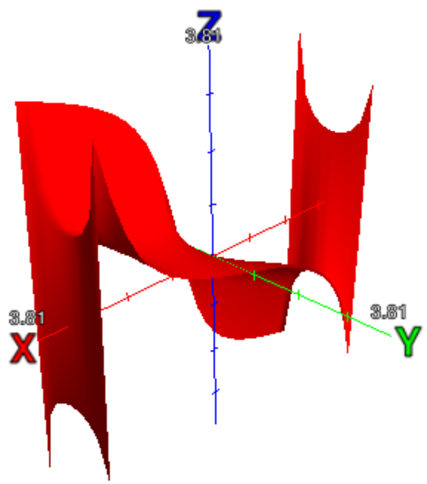
iptic function in the primitives

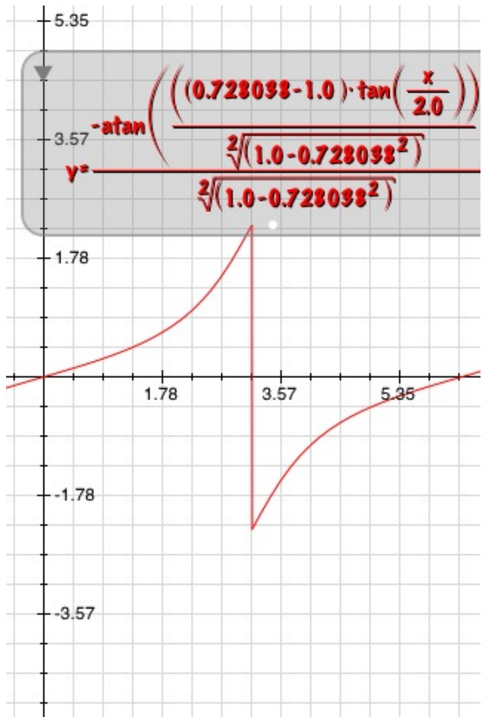
$$\frac{1}{\sqrt{e^2-1}}\tanh^{-1}\left(\frac{(e-1)\tan(\frac{\nu}{2})}{\sqrt{e^2-1}}\right)$$

ernate form for program computation

$$-\frac{1}{\sqrt{1-e^2}}\tan^{-1}\left(\frac{(e-1)\tan(\frac{\nu}{2})}{\sqrt{1-e^2}}\right)$$

aphs of the elliptic function





mber of characteristical primitives

or the third body problem

r the SRP problem

Integration of the Osculating Lagrange Planetary Equations (3)

nstants of the integrals

n integral is controled by :

e step of the elliptic function at the value of π of the true anomaly,

e value of zero of the integral at the interval lower bound.

uation of time

$$t = t_0 + \frac{2\tan^{-1}(\sqrt{\frac{1-e}{1+e}}\tan(\frac{v}{2})) - e\ \sin(2\tan^{-1}(\sqrt{\frac{1-e}{1+e}}\tan(\frac{v}{2}))) + \Delta(M)}{n}$$

nsistency

reduction of the interval bounds of the definite integrals allows to converge to the true ution.

Computation algorithm

nalytical step

ompute the integrals and solve the differential system of the Lagrange Planetary Equatior

ompute the equation of time.

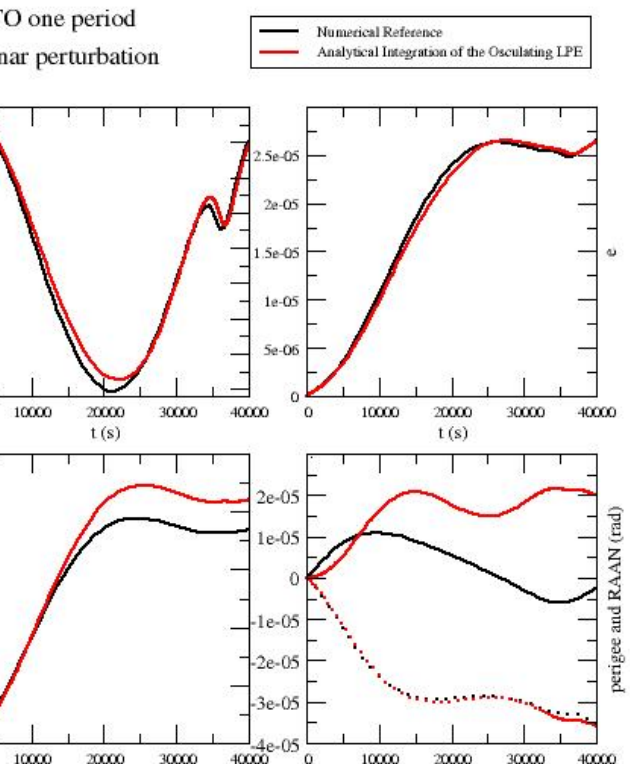
umerical step

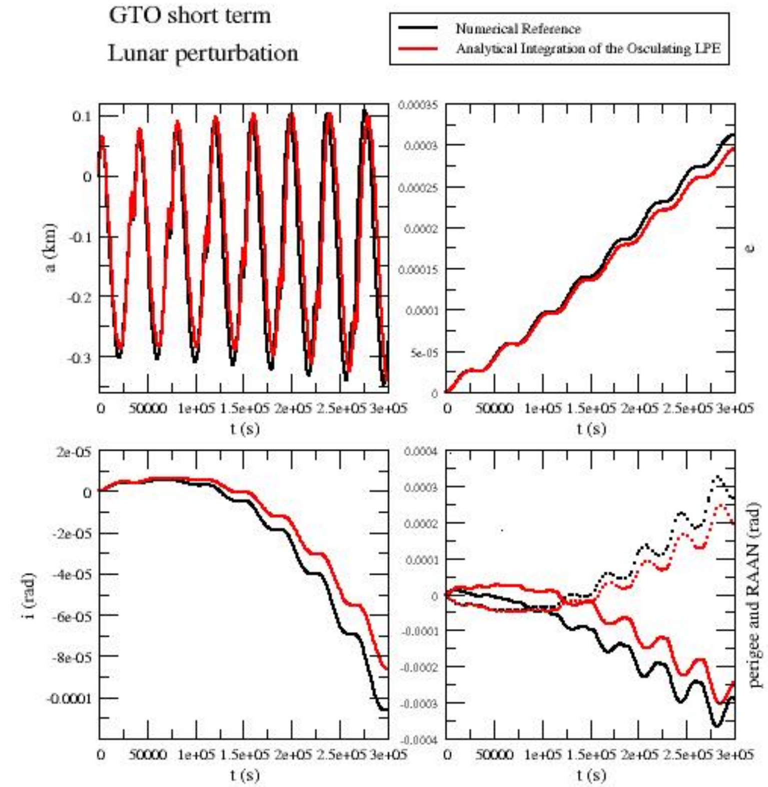
odate each keplerian element,

odate the position of the third body,

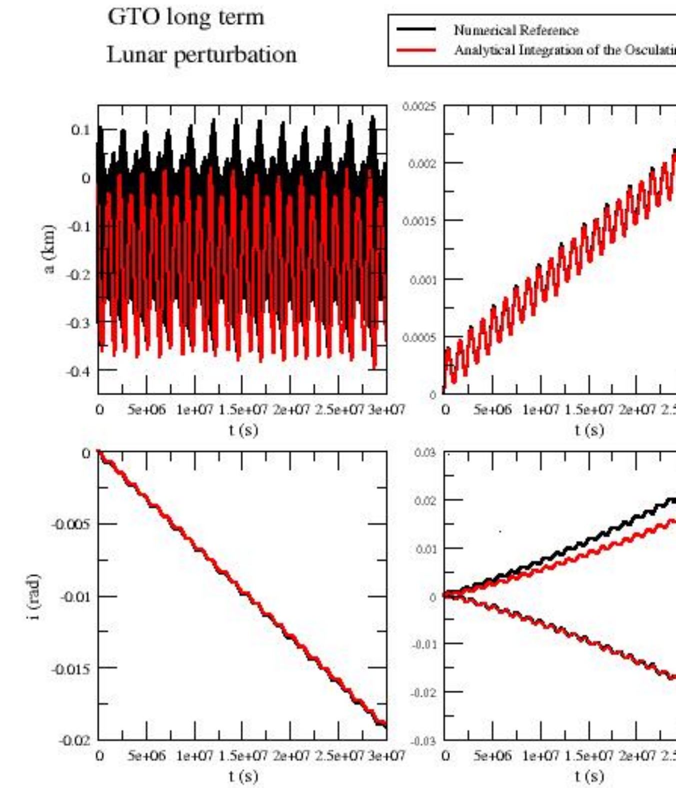
turn to the analytical step and so on.

eostationary Transfer Orbit lunar effects





Keplerian I.C. at t = 22028.777627314 CNES Julian Day in the VEIS 1950 quasi-inertial frame :
a = 0.243725807234344E+05 km, e = 0.728038503096540E+00, i =0.299655550675789E+01 deg,
perigee = 0.178399919209522E+03 deg, RAAN = -.121943444117876E+03 deg, M =17.6053049 deg : v = 0.981379935E+02 deg



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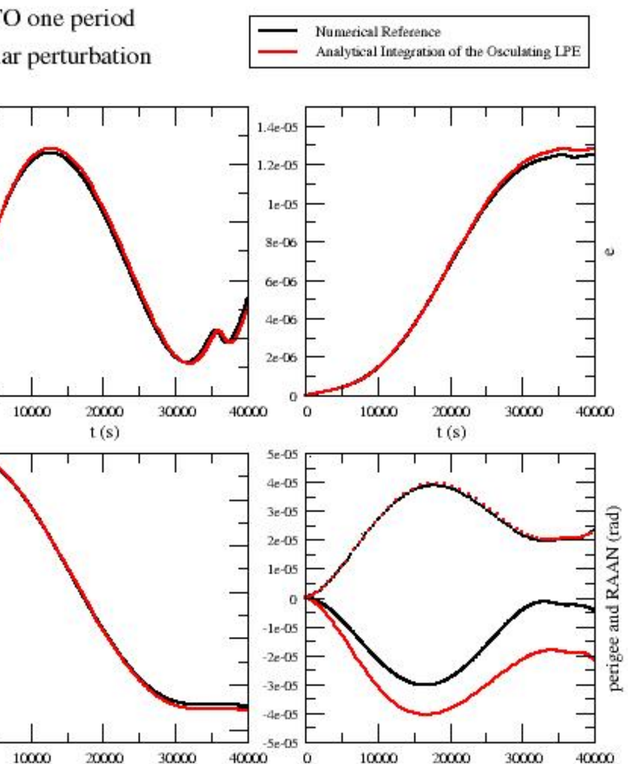

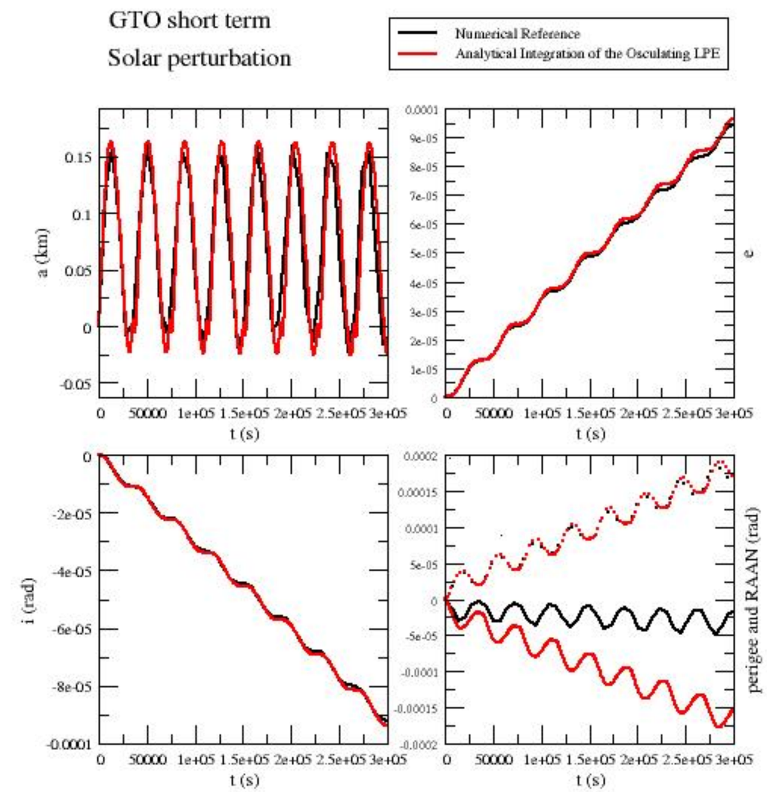

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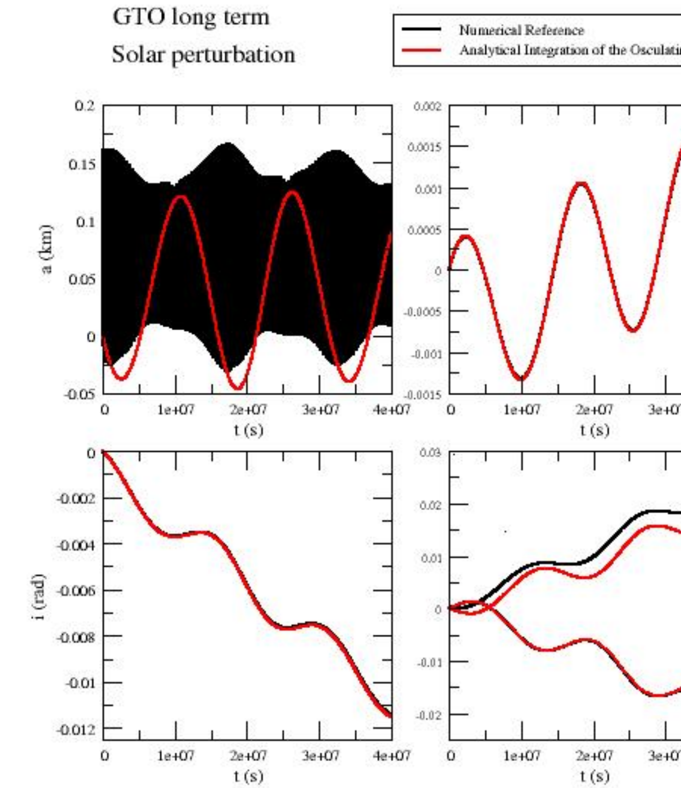

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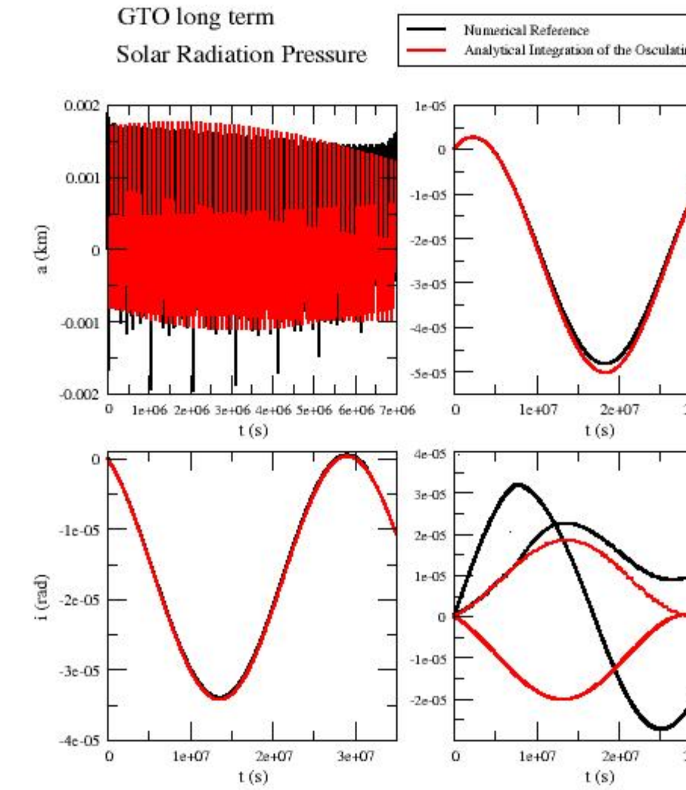
eostationary Transfer Orbit SRP effects



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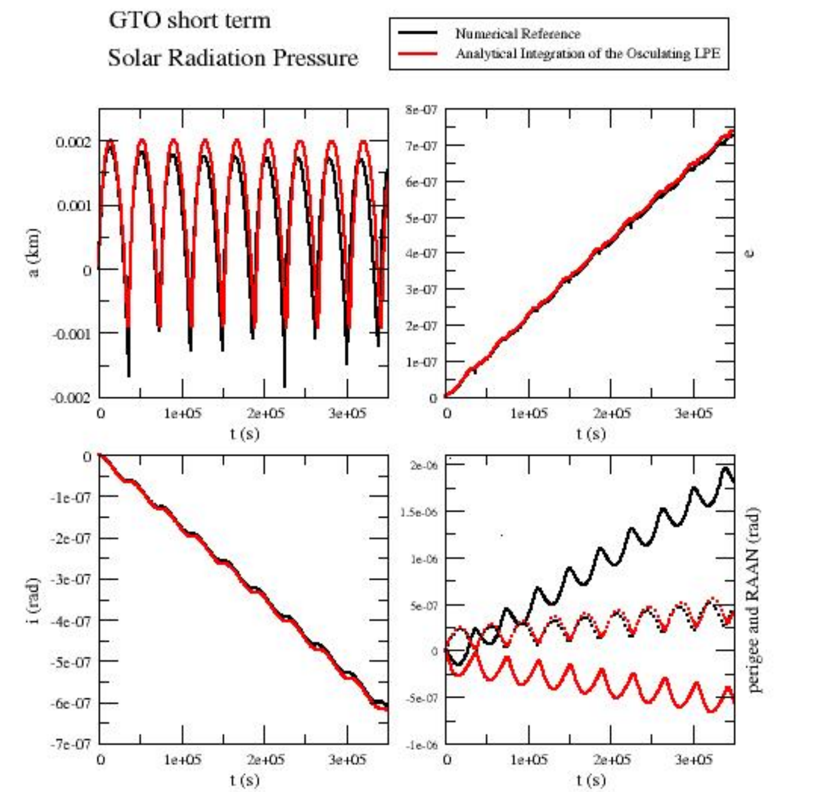

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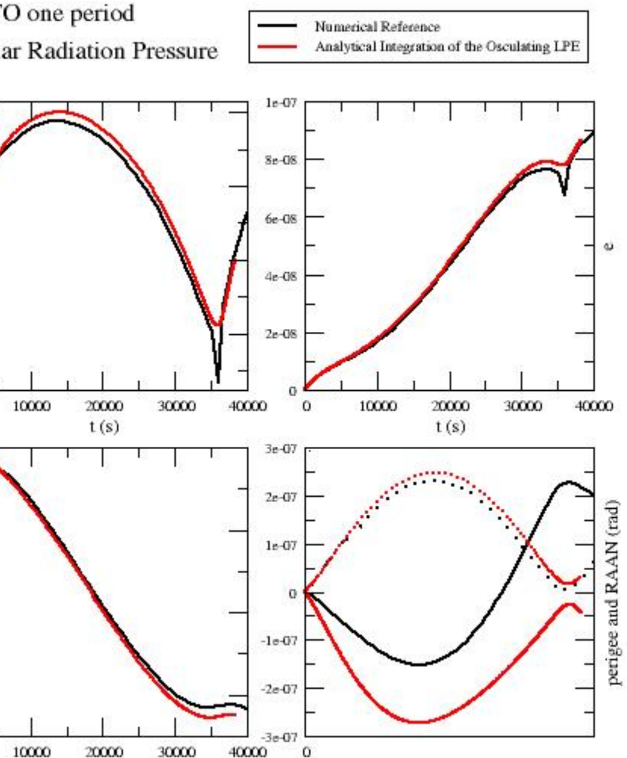
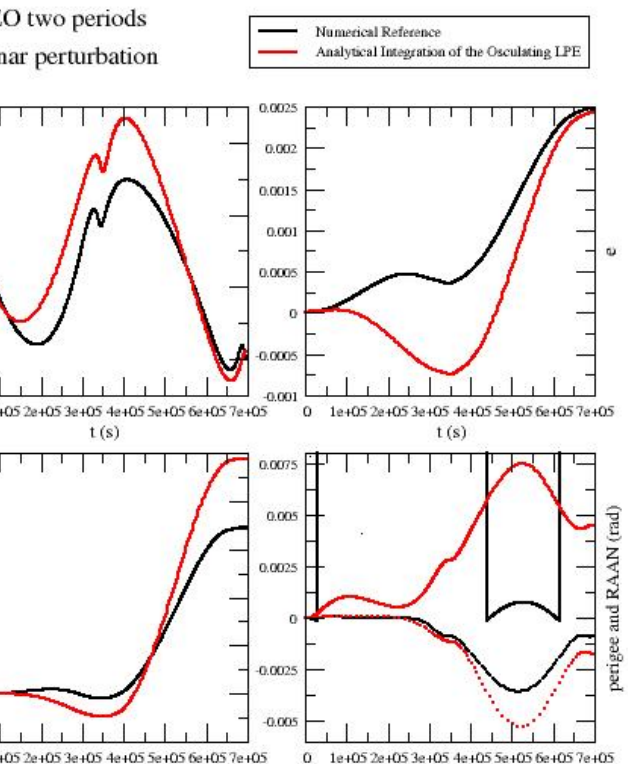
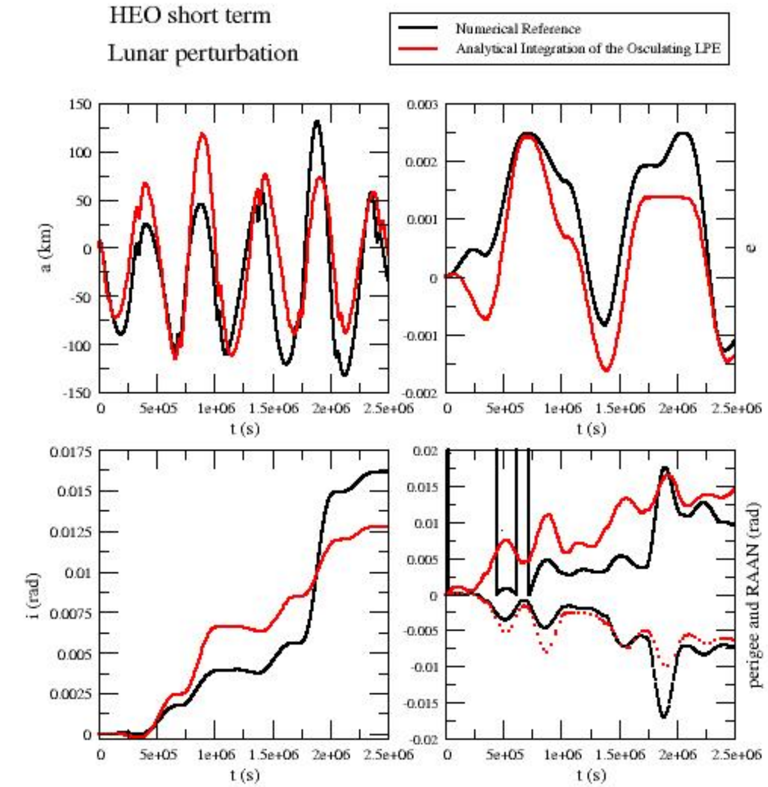
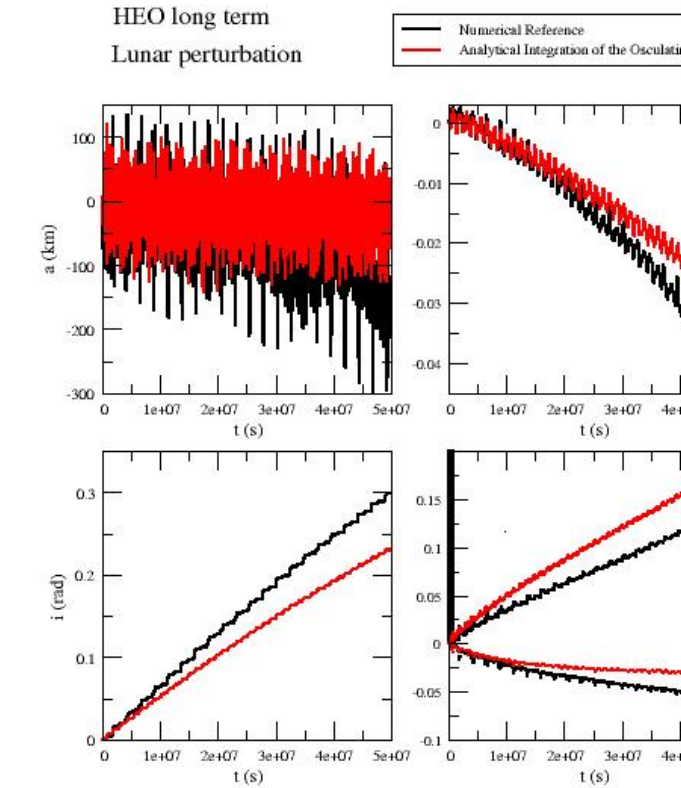


gh Elliptic Orbit lunar effects





Keplerian I.C. at t = 23557.86338 CNES Julian Day in the VEIS 1950 quasi-inertial frame :
a =106247.136454 km , e = 0.75173 i =5.2789 deg,
perigee = -179.992 deg, RAAN = 89.351 deg, M = 0.0 deg : v = 0.0 deg



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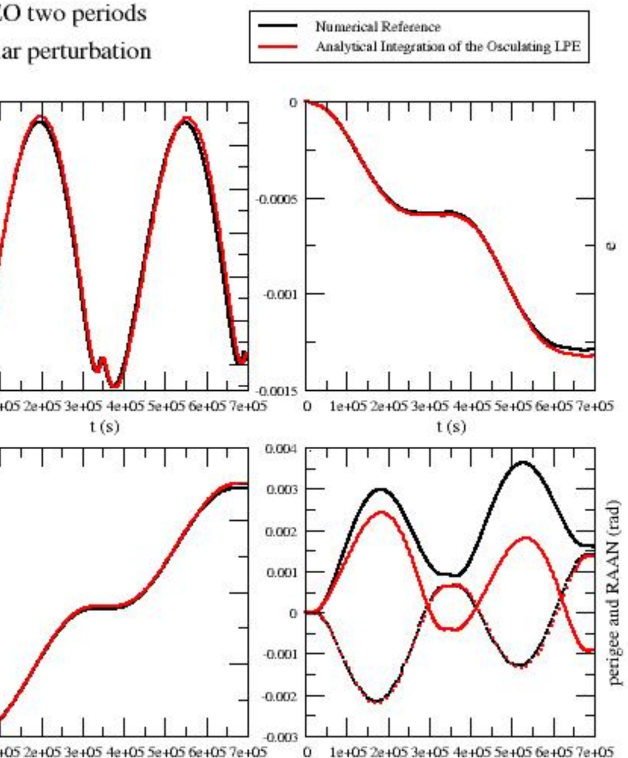
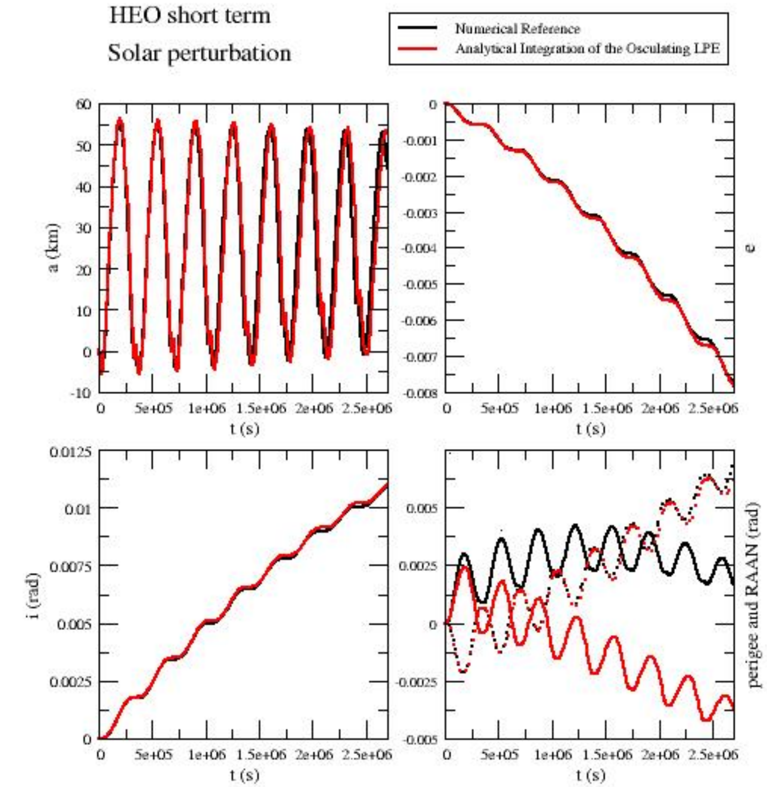
gh Elliptic Orbit solar effects





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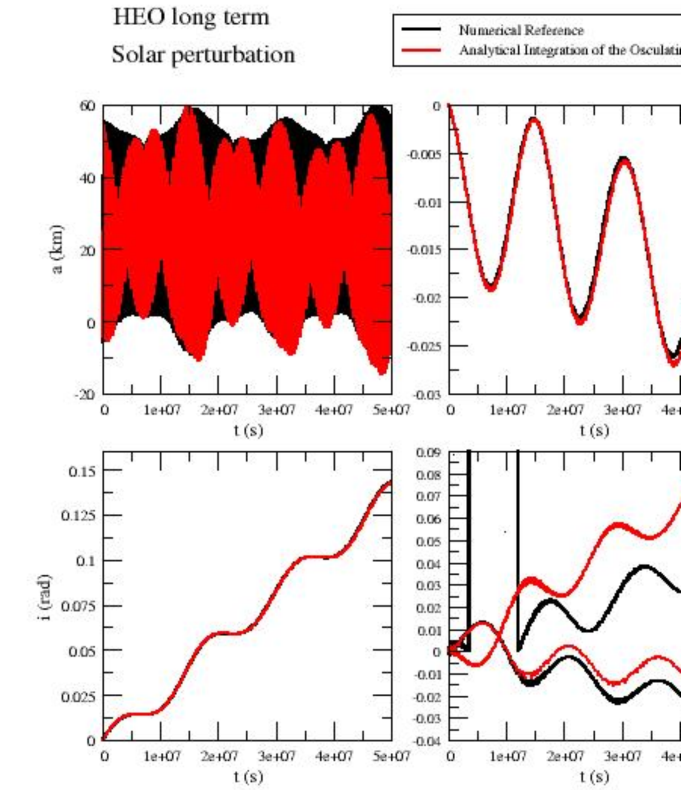

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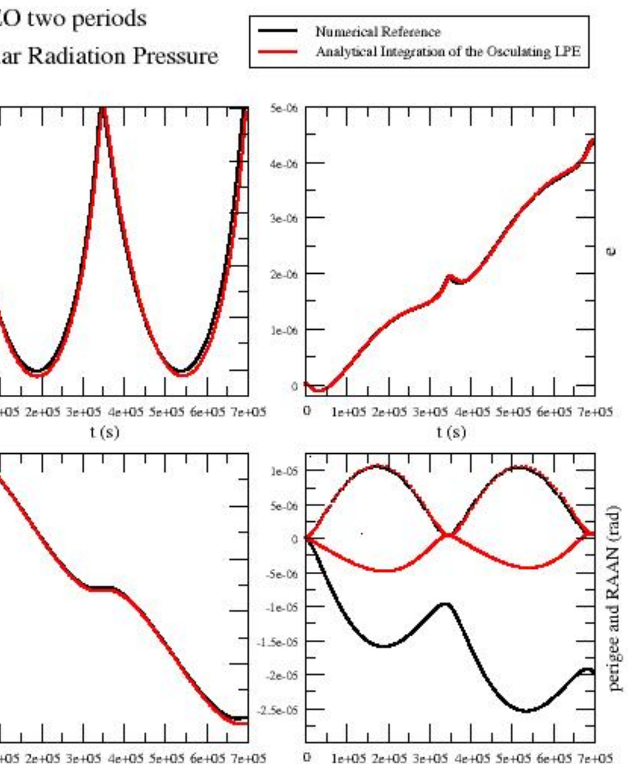
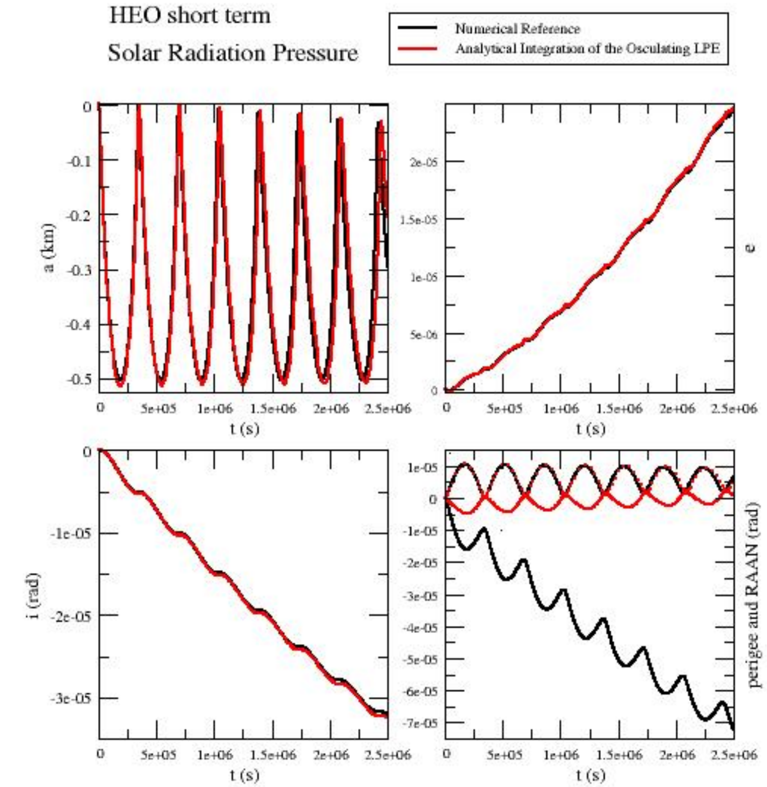
gh Elliptic Orbit SRP effects





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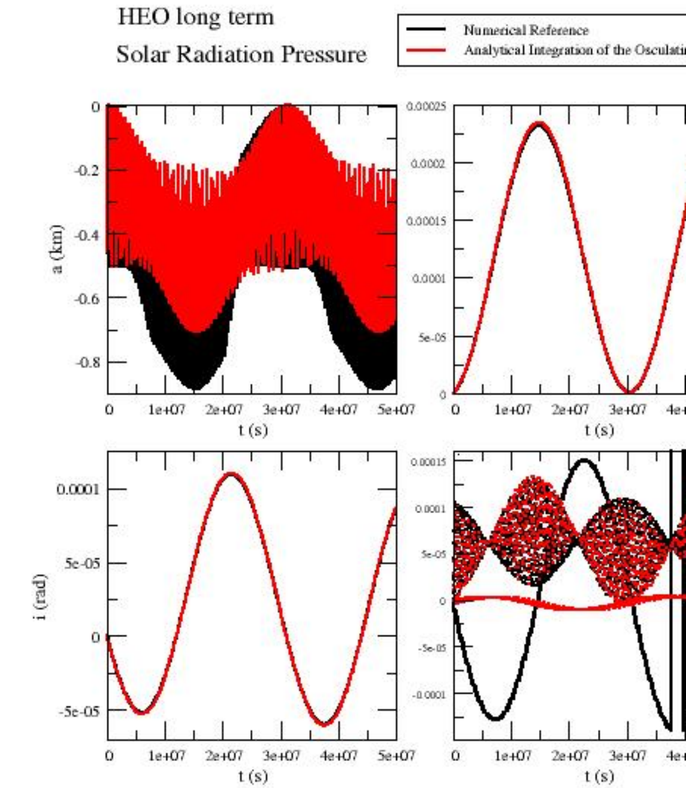

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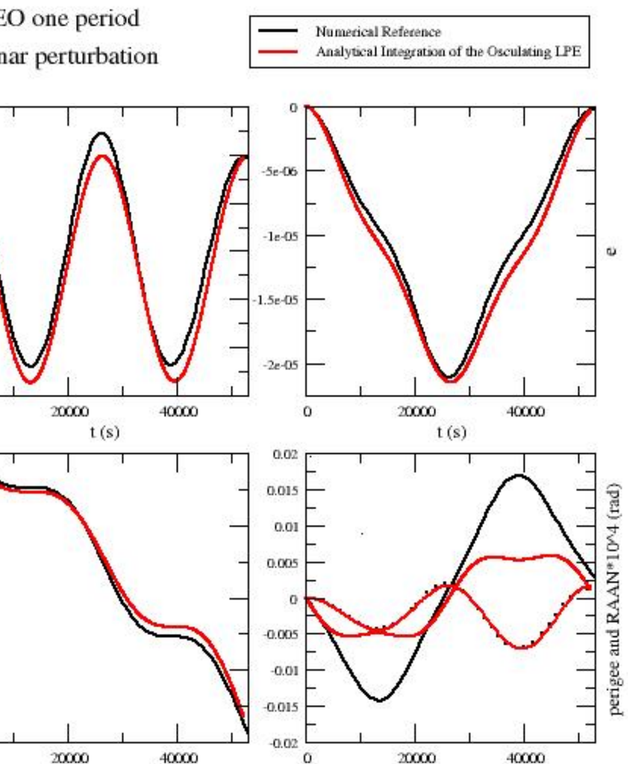
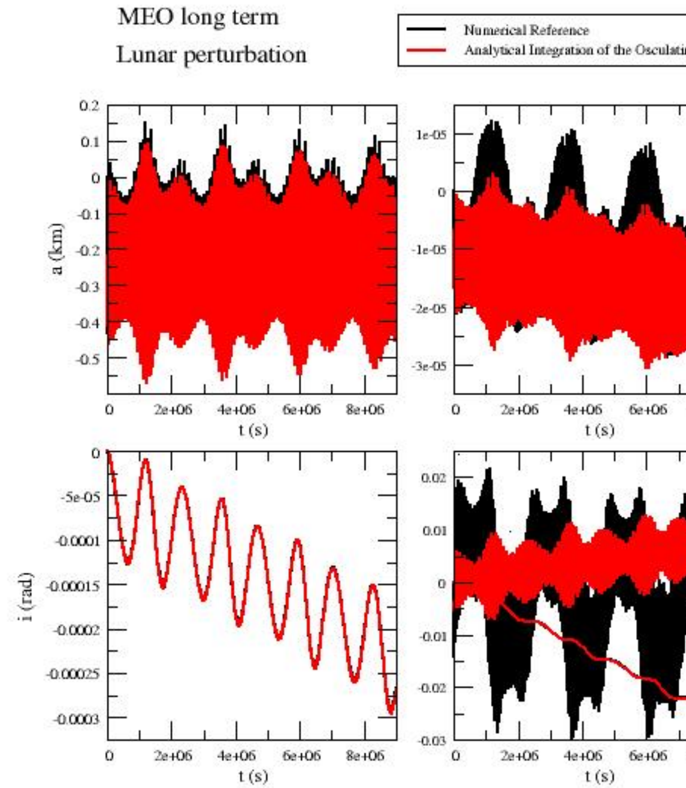
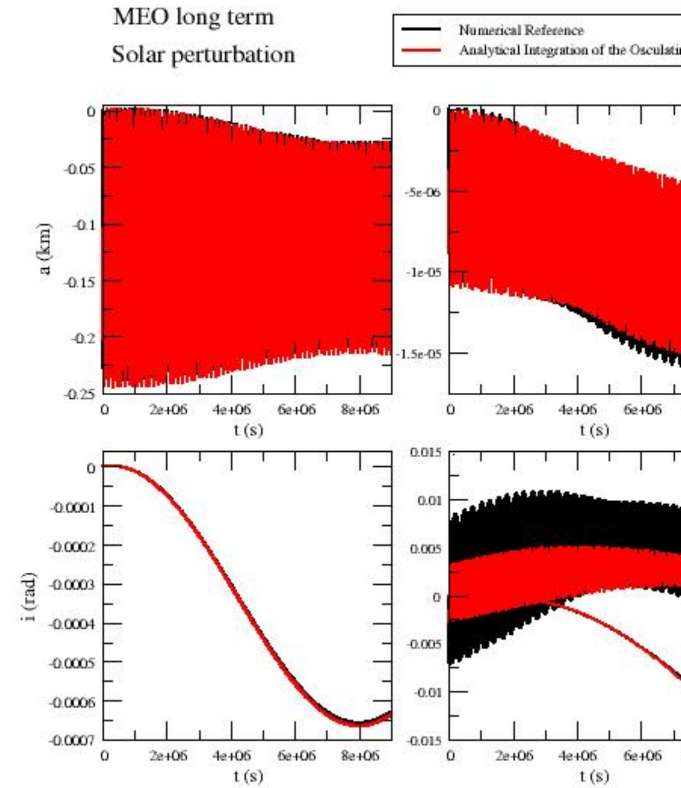




Keplerian I.C. at t = 17795.0 CNES Julian Day in the VEIS 1950 quasi-inertial frame :
a = 29995.225 km , e = 0.00104 i = 56.0 deg,
perigee = 0.0 deg, RAAN = 0.0 deg, M = 0.0 deg : v = 0.0 deg

ean Earth Orbit solar effects



Keplerian I.C. at t = 17795.0 CNES Julian Day in the VEIS 1950 quasi-inertial frame :
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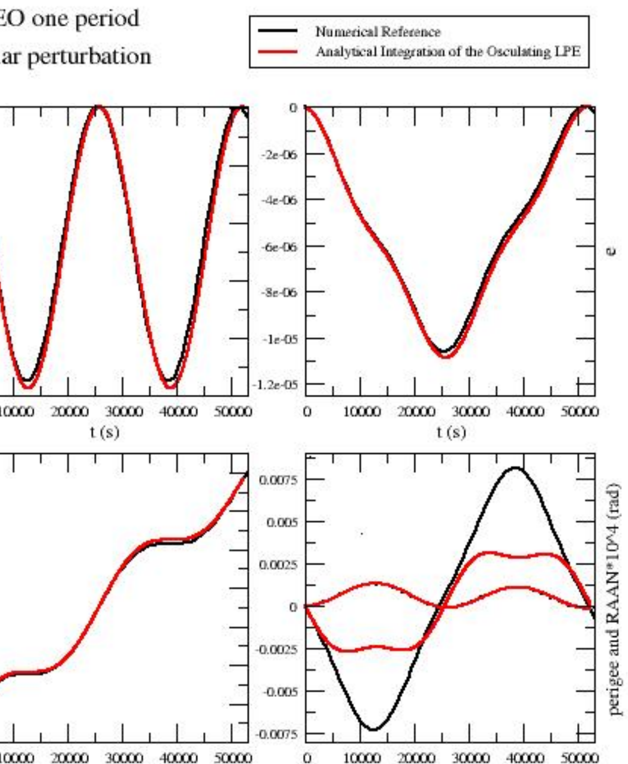
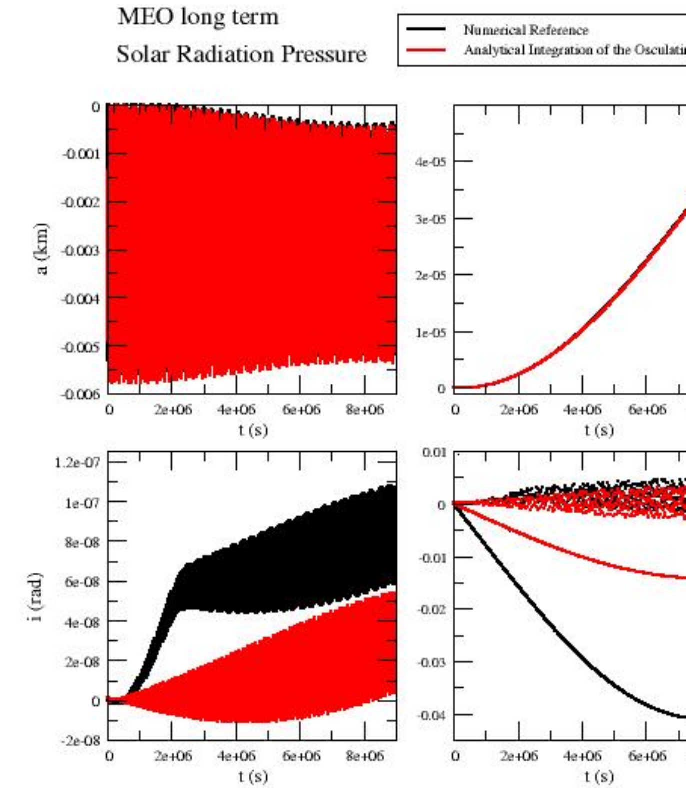
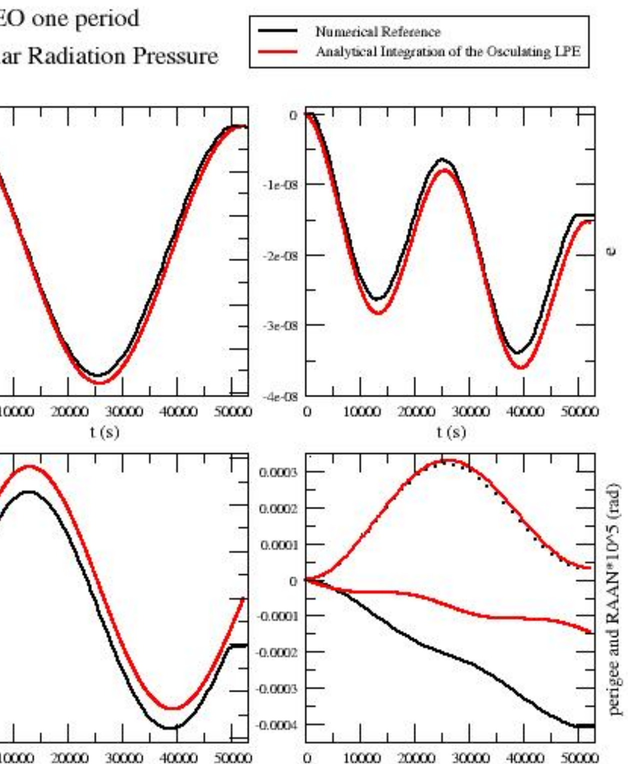


ean Earth Orbit SRP effects



Keplerian I.C. at t = 17795.0 CNES Julian Day in the VEIS 1950 quasi-inertial frame :
a = 29995.225 km , e = 0.00104 i = 56.0 deg,
perigee = 0.0 deg, RAAN = 0.0 deg, M = 0.0 deg : v = 0.0 deg



Results (10)

nderstanding

MPLIFIED MODELS OF THE POTENTIALS

CULATING INTEGRATION

ACT INTEGRATION WITHOUT ANY SIMPLIFYING ASSUMPTION

LIDITY FOR ALL THE ECCENTRICITIES EXCLUDING THE NULL VALUE

THEMATICAL CONSISTENCY OF THE METHOD

:

FFICULTY WITH THE PERIGEE EVEN WHEN THE DIFFERENTIAL SYSTEM IS WELL
ITEGRATED

E ACCURACY REDUCES WHEN THE ALTITUDE IS VERY HIGH

T OPTIMIZED COMPUTATION

ypothesis

XING PERIGEE AND MEAN ANOMALY

E REDUCED ACCURACY DUE TO THE SIMPLIFIED MODEL OF THE LUNAR POTENTIAL

TTER ACCURACY WITH AN OPTIMIZED COMPUTATION

Conclusion

he Osculating Lagrange Planetary Equations for simplified modelisation

s a simple method intellectually efficient to analyse each effect of the Sun and of the Moo
ver one orbital period. Moreover, a long duration simulation allows to observe the secula
ffects on the orbital parameters.

e osculating formulas could complement other modelisations.

e coupling with the gravitationnal first zonal term J_2 provides a simple conservative mode

reover, the numerical-analytical method allows to built a complete modelisation in Gener
erturbations.

o be continued

e order 3 of the third body potential is realizable.