Orbit determination with the two-body integrals

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Outline

- The orbit determination problem
- The integrals of Kepler's problem
- Optical observations and attributables
- Algebraic methods for preliminary orbits
- A test case
- Conclusions and open problems

The orbit determination problem: old and new

- **observations**: single observations per night
- **preliminary orbits**: Gauss or Laplace methods
- **least squares fit**: refinement of the preliminary orbits

- **observations**: very short arcs (VSAs)
- **identification of VSAs for preliminary orbits**: new methods needed
- **least squares fit**: still necessary

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{|\mathbf{r}|^3}\,, \qquad \mathbf{r} \in \mathbb{R}^3,\ \mu > 0.$$

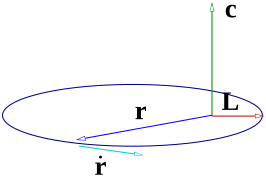
Kepler's problem

INTEGRALS OF MOTIONS

$$\mathbf{c} = \mathbf{r} \times \dot{\mathbf{r}}$$ *angular momentum*

$$\mathcal{E} = \frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{\mu}{|\mathbf{r}|}$$ *energy*

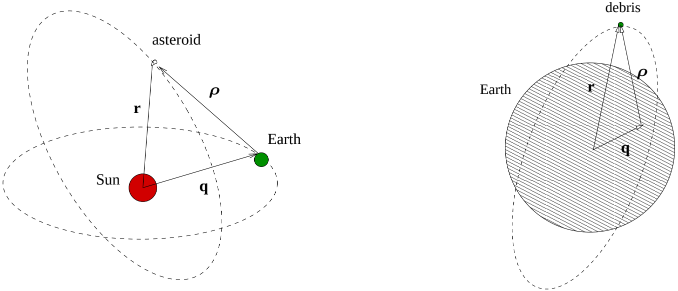
$$\mathbf{L} = \frac{1}{\mu}\dot{\mathbf{r}} \times \mathbf{c} - \frac{\mathbf{r}}{|\mathbf{r}|}$$ *Laplace-Lenz vector*



Only 5 scalar integrals are independent:

$$\mathbf{L} \cdot \mathbf{c} = 0$$
$$2|\mathbf{c}|^2\mathcal{E} + \mu^2(1 - |\mathbf{L}|^2) = 0$$

Asteroid and space debris observations

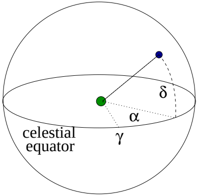


$$\mathbf{r} = \boldsymbol{\rho} + \mathbf{q}$$

$$\boldsymbol{\rho} = \rho \hat{\mathbf{e}}^{\rho} \qquad \hat{\mathbf{e}}^{\rho} = \text{observation direction}$$

To compute an orbit we have to determine $\mathbf{r}, \dot{\mathbf{r}}$ at some time.

Optical observations of a Solar system body



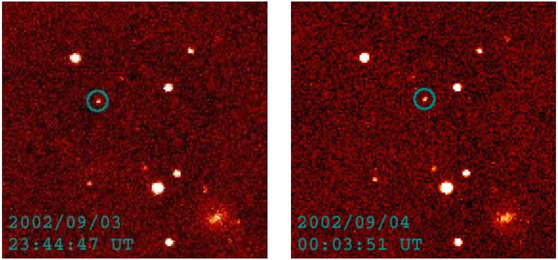
α=right ascension

δ= declination

An optical observation consists of two angles (α, δ) on the celestial sphere.

The topocentric distance ρ of the the observed body is unknown.

DETECTION OF AN ASTEROID



Optical attributables

Given a short arc of optical observations

$$(\alpha_i, \delta_i) \in S^1 \times (-\pi/2, \pi/2) \text{ at times } t_i,\ i = 1 \ldots m,$$

of a Solar system body we can compute an attributable

$$\mathcal{A} = (\alpha, \delta, \dot{\alpha}, \dot{\delta})$$

at time $\bar{t} = \frac{1}{m}\sum_i t_i$ by linear or quadratic interpolation.

The radial distance ρ and the radial velocity $\dot{\rho}$ are completely undetermined.

Algebraic methods for the linkage problem

Linkage/correlation: identification of two short arcs as related to the same observed body. Using the data of both arcs we can try to compute an orbit.
Our main idea is to perform linkage by solving polynomial equations using the first integrals of the Kepler problem. This allows a global control on the solutions.

Taff and Hall (1977): first suggested to use the integrals $\mathbf{c}, \mathcal{E}$ to write equations for the linkage problem.

Gronchi, Dimare and Milani (2010): proposed to solve the equations by Taff and Hall by algebraic methods.

Gronchi, Farnocchia and Dimare (2011): proposed to solve different equations, using a suitable projection of the Laplace-Lenz vector in place of the energy.

ANGULAR MOMENTUM:

$$\mathbf{c}(\rho, \dot{\rho}) = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{D}\dot{\rho} + \mathbf{E}\rho^2 + \mathbf{F}\rho + \mathbf{G}$$

with

$$\begin{aligned}
\mathbf{D} &= \mathbf{q} \times \hat{\mathbf{e}}^\rho \\
\mathbf{E} &= \dot{\alpha}\cos\delta \hat{\mathbf{e}}^\rho \times \hat{\mathbf{e}}^\alpha + \dot{\delta}\hat{\mathbf{e}}^\rho \times \hat{\mathbf{e}}^\delta \\
\mathbf{F} &= \dot{\alpha}\cos\delta \mathbf{q} \times \hat{\mathbf{e}}^\alpha + \dot{\delta}\mathbf{q} \times \hat{\mathbf{e}}^\delta + \hat{\mathbf{e}}^\rho \times \dot{\mathbf{q}} \\
\mathbf{G} &= \mathbf{q} \times \dot{\mathbf{q}}
\end{aligned}$$

where $\hat{\mathbf{e}}^\rho, \hat{\mathbf{e}}^\alpha, \hat{\mathbf{e}}^\delta$ are unit vectors depending only on α, δ.
Thus $\mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}$ depend only on the attributable $\mathcal{A}$ and on the motion of the observer $\mathbf{q}, \dot{\mathbf{q}}$ at time $\bar{t}$.

Keplerian integrals as function of $(\rho, \dot\rho)$

ENERGY:

$$\mathcal{E} = \tfrac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{\mu}{|\mathbf{r}|}$$

LAPLACE-LENZ VECTOR:

$$\mu\mathbf{L}(\rho,\dot\rho) = \dot{\mathbf{r}} \times \mathbf{c} - \mu\frac{\mathbf{r}}{|\mathbf{r}|} = \left(|\dot{\mathbf{r}}|^2 - \frac{\mu}{|\mathbf{r}|}\right)\mathbf{r} - (\dot{\mathbf{r}}\cdot\mathbf{r})\dot{\mathbf{r}}$$

where

$$\begin{aligned}
|\mathbf{r}| &= (\rho^2 + |\mathbf{q}|^2 + 2\rho\mathbf{q}\cdot\hat{\mathbf{e}}^\rho)^{1/2}\,,\\
|\dot{\mathbf{r}}|^2 &= \dot\rho^2 + (\dot\alpha^2\cos^2\delta + \dot\delta^2)\rho^2 + 2\dot{\mathbf{q}}\cdot\hat{\mathbf{e}}^\rho\dot\rho +\\
&\quad +2\dot{\mathbf{q}}\cdot(\dot\alpha\cos\delta\hat{\mathbf{e}}^\alpha + \dot\delta\hat{\mathbf{e}}^\delta)\rho + |\dot{\mathbf{q}}|^2\\
\dot{\mathbf{r}}\cdot\mathbf{r} &= \rho\dot\rho + \mathbf{q}\cdot\hat{\mathbf{e}}^\rho\dot\rho + (\dot{\mathbf{q}}\cdot\hat{\mathbf{e}}^\rho + \mathbf{q}\cdot\hat{\mathbf{e}}^\alpha\dot\alpha\cos\delta + \mathbf{q}\cdot\hat{\mathbf{e}}^\delta\dot\delta)\rho + \dot{\mathbf{q}}\cdot\mathbf{q}
\end{aligned}$$

Equating the angular momentum

Given two attributables $\mathcal{A}_1, \mathcal{A}_2$ at times $\bar{t}_1, \bar{t}_2$, equating the angular momentum at the two times we obtain

$$\mathbf{D}_1\dot{\rho}_1 - \mathbf{D}_2\dot{\rho}_2 = \mathbf{J}(\rho_1, \rho_2) \tag{1}$$

where

$$\mathbf{J}(\rho_1, \rho_2) = \mathbf{E}_2\rho_2^2 - \mathbf{E}_1\rho_1^2 + \mathbf{F}_2\rho_2 - \mathbf{F}_1\rho_1 + \mathbf{G}_2 - \mathbf{G}_1 \ .$$

By scalar multiplication of (1) with $\mathbf{D}_1 \times \mathbf{D}_2$ we perform elimination of the variables $\dot{\rho}_1, \dot{\rho}_2$; this yields

$$q(\rho_1, \rho_2) \stackrel{def}{=} \mathbf{D}_1 \times \mathbf{D}_2 \cdot \mathbf{J}(\rho_1, \rho_2) = 0$$

Equating the energy

By vector multiplication of (1) with $\mathbf{D}_1$ and $\mathbf{D}_2$, projecting on $\mathbf{D}_1 \times \mathbf{D}_2$ we obtain

$$\dot{\rho}_1(\rho_1,\rho_2) = \frac{(\mathbf{J}\times\mathbf{D}_2)\cdot(\mathbf{D}_1\times\mathbf{D}_2)}{|\mathbf{D}_1\times\mathbf{D}_2|^2}\,; \qquad \dot{\rho}_2(\rho_1,\rho_2) = \frac{(\mathbf{J}\times\mathbf{D}_1)\cdot(\mathbf{D}_1\times\mathbf{D}_2)}{|\mathbf{D}_1\times\mathbf{D}_2|^2}$$

Substituting into $\mathcal{E}_1 = \mathcal{E}_2$, rearranging the terms and squaring twice we obtain a polynomial equation

$$p(\rho_1,\rho_2) = 0$$

with total degree 24.

Intersections between the curves

We study the semi-algebraic problem

$$\left\{ \begin{array}{l} p(\rho_1, \rho_2) = 0 \\ q(\rho_1, \rho_2) = 0 \end{array} \right. , \qquad \rho_1, \rho_2 > 0 \tag{2}$$

by resultant theory.

We can write

$$p(\rho_1, \rho_2) = \sum_{j=0}^{20} a_j(\rho_2)\, \rho_1^j \,,$$
$$q(\rho_1, \rho_2) = b_2\, \rho_1^2 + b_1\, \rho_1 + b_0(\rho_2)$$

for some coefficients a_i, b_j, depending only on ρ_2.

Elimination of ρ_1

Consider the resultant $Res(\rho_2)$ of p, q with respect to ρ_1: it is a 48 degree polynomial defined as the determinant of the Sylvester matrix

$$\begin{pmatrix} a_{20} & 0 & b_2 & 0 & \dots & \dots & 0 \\ a_{19} & a_{20} & b_1 & b_2 & 0 & \dots & 0 \\ \vdots & \vdots & b_0 & b_1 & b_2 & \dots & \vdots \\ \vdots & \vdots & 0 & b_0 & b_1 & \dots & \vdots \\ a_0 & a_1 & \vdots & \vdots & \vdots & b_0 & b_1 \\ 0 & a_0 & 0 & 0 & 0 & 0 & b_0 \end{pmatrix} .$$

The ρ_2 component of a solution of (2) must be a root of $Res(\rho_2)$.

Computation of the solutions

- Compute the positive roots ρ_2 of $Res(\rho_2)$.
- For each root find the corresponding values of ρ_1, $\dot{\rho}_1, \dot{\rho}_2$.
- Discard spurious solutions, obtained by squaring.
- Compute the related Keplerian orbits at times

$$\tilde{t}_i = \bar{t}_i - \frac{\rho_i}{c}, i = 1, 2$$

corrected by aberration.

Selecting the solutions I: compatibility conditions

The knowledge of the angular momentum vector and of the energy at a given time yields the values of

$$a, e, I, \Omega \; .$$

From $\mathbf{c}_1 = \mathbf{c}_2$, $\mathcal{E}_1 = \mathcal{E}_2$ we obtain the same values of a, e, I, Ω at times $\tilde{t}_1, \tilde{t}_2$, but we must check for the *compatibility conditions*

$$\omega_1 = \omega_2 \,, \qquad \ell_1 = \ell_2 + n(\tilde{t}_1 - \tilde{t}_2) \,, \tag{3}$$

where n is the *mean motion*.

Problem: how to take into account the errors in the observations?

Consider the map

$$(\mathcal{A}_1, \mathcal{A}_2) \overset{\Psi}{\mapsto} (\mathcal{A}_1, \rho_1, \dot{\rho}_1, \Delta_{1,2}) \ , \qquad \Delta_{1,2} = (\Delta\omega, \Delta\ell)$$

giving the orbit at time $\tilde{t}_1$, and the difference in ω, ℓ.

Check whether $\Delta_{1,2} = 0$ is compatible with the observational errors by covariance propagation through the map Ψ.

This algorithm also allows to define covariance matrices for the preliminary orbits that we compute.

Equating a component of the Laplace-Lenz vector

It is possible to reduce the algebraic degree of the linkage problem by writing different equations (i.e. using different integrals). In Gronchi, Farnocchia, Dimare (2011) we select a suitable component of the Laplace-Lenz vector in place of the energy.

Given $\mathcal{A}_1, \mathcal{A}_2$ we equate $\mathbf{L}_1, \mathbf{L}_2$ projected along $\mathbf{v} = \hat{\mathbf{e}}_2^{\rho} \times \mathbf{q}_2$:

$$\mathbf{L}_1(\rho_1, \dot{\rho}_1) \cdot \mathbf{v} = \mathbf{L}_2(\rho_2, \dot{\rho}_2) \cdot \mathbf{v} \; .$$

Equating a component of the Laplace-Lenz vector

We have

$$\left(|\dot{\mathbf{r}}_1|^2 - \frac{\mu}{|\mathbf{r}_1|}\right)(\mathbf{r}_1 \cdot \mathbf{v}) - (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1)(\dot{\mathbf{r}}_1 \cdot \mathbf{v}) = -(\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)(\dot{\mathbf{r}}_2 \cdot \mathbf{v}) \, .$$

Rearranging the terms and squaring we obtain

$$\tilde{p}(\rho_1, \rho_2) \stackrel{def}{=} \mu^2(\mathbf{r}_1 \cdot \mathbf{v})^2 - |\mathbf{r}_1|^2 \left\{ \left[|\dot{\mathbf{r}}_1|^2 \mathbf{r}_1 - (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1)\dot{\mathbf{r}}_1 + (\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)\dot{\mathbf{r}}_2 \right] \cdot \mathbf{v} \right\}^2 = 0 \, .$$

$\tilde{p}$ is a polynomial of total degree 10 in ρ_1, ρ_2, therefore

$$\tilde{p}(\rho_1, \rho_2) = 0 \, , \qquad q(\rho_1, \rho_2) = 0 \, , \qquad \rho_1, \rho_2 > 0 \, .$$

has degree 20.

Selecting the solutions II: attribution

We can select the solutions of the linkage problem also by means of the attribution algorithm.

Let $\mathcal{E}_1$ be a set of orbital elements at time t_1, with covariance matrix Γ_1. Propagate orbit and covariance to the epoch $\bar{t}_2$ of $\mathcal{A}_2$, with covariance $\Gamma_{\mathcal{A}_2}$. Then extract a predicted attributable $\mathcal{A}_p$, at time $\bar{t}_2$, with covariance $\Gamma_{\mathcal{A}_p}$.

We can compare $\mathcal{A}_p, \Gamma_{\mathcal{A}_p}$ with $\mathcal{A}_2$, $\Gamma_{\mathcal{A}_2}$ by defining an identification penalty χ_4, that gives a measure of the '*price to pay*' assuming the observations of both attributables belong to the same celestial body.

A test case: (99942) Apophis

DATA: two sets of observations with mean epochs $\bar{t}_1 = 53175.59$, $\bar{t}_2 = 53357.45$ MJD.
SOLUTIONS:

	ρ_1	ρ_2
1	0.78987	0.04345
2	1.13777	0.09569

Attribution: $\chi_4(1) = 3230925.94$, $\chi_4(2) = 2.29$. We select the second one:

$$a = 0.923\,, \quad e = 0.189\,, \quad I = 3.287$$
$$\Omega = 204.912\,, \quad \omega = 124.778\,, \quad \ell = 249.003$$

at epoch $t_1 = 53175.59$. We compare it with the known orbit:

$$a = 0.922\,, \quad e = 0.191\,, \quad I = 3.333$$
$$\Omega = 204.575\,, \quad \omega = 126.176\,, \quad \ell = 247.500$$

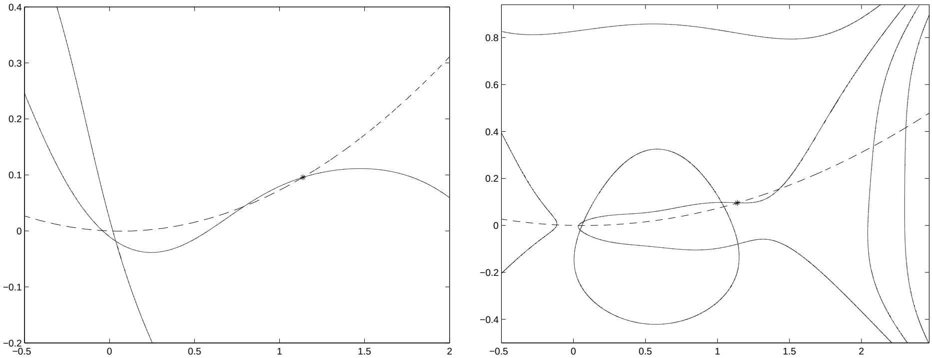


Figure: Test case with (99942) Apophis. Left: with Laplace-Lenz vector. Right: with energy.

In Dimare et al. 2011 a comparison of the two methods is shown for a LEO population.

Conclusions and open problems

The investigation of new orbit determination methods is essential to deal with modern observations. Algebraic methods are suitable and effective for the linkage problem.

J_2 EFFECTS: in case of space debris the effect of oblateness of the Earth cannot be neglected: in Farnocchia et al. (2010) a solution is proposed.

RADAR OBSERVATIONS: We can define a radar attributable $\mathcal{A}_{rad} = (\alpha, \delta, \rho, \dot{\rho})$ at time $\bar{t}$. Here $\dot{\alpha}$, $\dot{\delta}$ remain completely undetermined.
Problem: large uncertainties of the angles α, δ. Can we use this data type?
If yes then we have an explicit solution of the linkage problem for the mixed case of 1 optical and 1 radar attributable.

References

[1] **L. Dimare et al.** *Innovative system of very wide field optical sensors for space surveillance in the LEO region*, AMOS conf. 2011

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[3] **G. F. Gronchi, L., Dimare, A., Milani**: *Orbit determination with the two-body integrals*, CMDA, 2010

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