ISSN 1560-3547, Regular and Chaotic Dynamics, 2009, Vol. 14, No. 2, pp. 179–217. © Pleiades Publishing, Ltd., 2009.

The Hamiltonian Dynamics of Self-gravitating Liquid and Gas Ellipsoids

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Received August 3, 2008; accepted December 1, 2008

Abstract—The dynamics of self-gravitating liquid and gas ellipsoids is considered. A literary survey and authors' original results obtained using modern techniques of nonlinear dynamics are presented. Strict Lagrangian and Hamiltonian formulations of the equations of motion are given; in particular, a Hamiltonian formalism based on Lie algebras is described. Problems related to nonintegrability and chaos are formulated and analyzed. All the known integrability cases are classified, and the most natural hypotheses on the nonintegrability of the equations of motion in the general case are presented. The results of numerical simulations are described. They, on the one hand, demonstrate a chaotic behavior of the system and, on the other hand, can in many cases serve as a numerical proof of the nonintegrability (the method of transversally intersecting separatrices).

MSC2000 numbers: 70Hxx

DOI: 10.1134/S1560354709020014

Key words: liquid and gas self-gravitating ellipsoids, integrability, chaotic behavior

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This review article is dedicated to the dynamics of self-gravitating liquid and gas ellipsoids. This area of mechanics is represented by extensive studies, which were in many cases carried out independently by scolars from different countries and published in various journals, not always readily available. Certain results in this area are still controversial. We make here an attempt of a systematic exposition of fundamental, both classical and fairly recent, results concerning the dynamics of ellipsoidal figures. We also present our fresh results demonstrating a chaotic behavior of oscillating fluid ellipsoids. Some unresolved problems are formulated.

In the first part of the article, we describe, in a closed, systematic form, basic results in the dynamics of ellipsoidal figures of ideal, incompressible fluid, starting from the foundational works of Dirichlet and Riemann. As is known, these studies were preceded by a period of investigations of the static equilibrium of a rotating fluid mass, which were dictated by the scientific interest in the figure of the Earth and traced back to Newton's *Mathematical Principles of Natural Philosophy* and works by Clairaut and Maclaurin. A brief historical survey of this period (also associated with the names of Jacobi, Mayer, and Liouville) can be found in a monograph by Chandrasekhar [1]. Fundamental discoveries by Dirichlet [2] and Riemann [3] (1857–1861) were a turning point in developing the theory of figures of equilibrium; Dirichlet and Riemann were the first to investigate the dynamics of fluid ellipsoids. They noted the existence of a finite-dimensional solution of the Euler equations of the ideal-fluid dynamics according to which the ellipsoid preserves its shape but deforms. This is an exact hydrodynamic solution, so that the question of investigating the dynamics of such fluid ellipsoids can be correctly posed.

We describe here the basic results of classical studies in a brief, closed form. We present the equations of motion in various representations and the integrals of motion; we also discuss the Lagrangian and Hamiltonian forms of the equations, which makes it possible to naturally carry out a symmetry-group-based reduction of the system. Next, we analyze partial solutions starting from the standard configurations found for the first time by Maclaurin, Jacobi, and Riemann. We note a new class of chaotic motions of ellipsoids in the form of irregular pulsations, with the axes of the ellipsoid remaining motionless in the absolute space. In this case, we numerically find periodic solutions, construct separatrices, and therefore present a computer proof of nonintegrability of the system in the general case. To analytically and numerically investigate such motions, we use the regularization of the equations of motion and formulate the hypothesis of the nonexistence of analytical integrals. A particular case of this hypothesis is the problem (tracing back to Riemann) of the nonintegrability of a geodesic flow on a very simple two-dimensional cubical surface in $\mathbb{R}^3$ (a cubic, in contrast to quadrics, quadratic surfaces).

In the second part of the article, we consider problems of the dynamics of a gas cloud with an ellipsoidal stratification. The account of initial results on equilibrium and stability of compressible liquid and gas masses was given by Jeans in his fundamental essay [4], to which the Adams Prize for 1917 was awarded. This book also considers possible applications of these studies to the problems of cosmogony. Note (see p. 147 of [4]) that the most elementary model which is different from the classical incompressible model of figures of equilibrium was given by Roche [5], who largely utilized this model in construction of his cosmogonic theory generalizing the Laplace hypothesis. The Roche model (as we call it following [4]) contains a gravitating Newtonian center surrounded by weightless atmosphere inside which self-gravitation is neglected. In this part, in a unified form, we present results obtained by Ovsyannikov, Dyson, Linden-Bell, Zel'dovich, Fujimoto, etc. Various forms of the equations of motion of the gas cloud are derived under certain thermodynamic assumptions. Some questions of the Lagrangian–Hamiltonian formalism are also discussed. A generalized Dedekind law of reciprocity is formulated. An analog of the Riemann equation for the case of a compressible fluid is obtained. A particular case of the expansion of an ellipsoidal cloud of ideal monoatomic gas in the absence of gravitation is analyzed in detail. This case was recently considered by Gaffet, who noted new integrals of this system. We formulate Gaffet's results more accurately, which enables us to establish the Liouvillian integrability in an extended space (Gaffet found integrals of the system presented, which can be obtained by a reduction with use of a nonautonomous Jacobi-type integral). In particular, the involutivity of all the Gaffet integrals is revealed. An analogy between the general system considered by Gaffet and the generalized Euler–Calogero system is noted. The question of the Lax representation in this case is discussed.

In conclusion, we present new partial solutions that are axisymmetric with or without the presence of gravitation. For the case without gravitation, the system is reduced to quadratures at arbitrary initial conditions. For the presence of gravitation, chaotic motions are noted, which is due to the nonintegrability of this system. Chaotic motions are also revealed in more general cases of oscillating gaseous ellipsoids.

I. DYNAMICS OF A SELF-GRAVITATING FLUID ELLIPSOID

1. INTRODUCTION

The studies by L. Dirichlet in the dynamics of a self-gravitating fluid ellipsoid are dated back to 1856–1857. Ditichlet reported these studies in his lectures in 1857 and simultaneously in the *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen* as a brief note [6]. Unfortunately, he had no time to describe and publish his results in full (due to his illness and untimely death in 1859). These studies were prepared for publication and posthumously published by Dedekind in 1861 [2].

Three basic results can be isolated in Dirichlet's study:

1. A new partial solution of the hydrodynamic equations is presented, which describes the motion of a homogeneous, self-gravitating ellipsoid, and the equations of motion (of fluid particles) in motionless axes are derived.
2. Seven first integrals of the obtained equations are found; six of them, linear in velocities, correspond to the conservation laws of vorticity and total momentum, and the seventh integral is the total energy of the moving fluid.
3. The motion of an axisymmetric ellipsoid is integrated in quadratures with the inclusion of Newton's and Maclaurin's spheroids as partial solutions (in this case, Dirichlet also analyzes the possibility of existence of the solutions found in the case of no external pressure, i. e., in vacuum).

It is interesting to note that Dirichlet noted the integrals corresponding to the conservation of the vorticity vector prior to the publication of the well-known study of 1858 by Helmholtz. [7]. As can be judged by the form of the obtained integrals, Dirichlet was aware (before Helmholtz) of the conservation of vorticity not only for a particular solution but also for the general hydrodynamic equations (Dirichlet's note [6] is also evidence for his awareness). This fact was also noted by Klein in his well-known lectures [8].

Dedekind, while preparing Dirichlet's results for publication, discovered the reciprocity law according to which each solution of the Ditrichlet equations is corresponded with a reciprocal solution in which the variables that describe the rotation of the ellipsoid and the fluid motion inside it are permutated; in particular, he presented a solution (the Dedekind ellipsoid) reciprocal to the Jacobi ellipsoid, with the coordinate axes remaining motionless in space and with the fluid moving inside this invariable region [9].

An enormous contribution to the investigation of the dynamics of the fluid ellipsoid was made by an outstanding work by Riemann [3], which appeared in 1861, virtually immediately after the publication of Dirichlet's studies. The basic results of this work can be briefly formulated as follows:

1. The equations of motion in moving axes (the principal axes of the ellipsoid) were obtained, so that the order of the system was lowered and a linear-integral-based reduction was done. Furthermore, Riemann represented the equations of motion of the reduced system in a Hamiltonian form with a linear Lee–Poisson bracket (Riemann himself called this procedure reduction to a better observable form).
2. All partial solutions corresponding to the motion of the ellipsoid without changes in its form were presented and conditions of their existence were analyzed (i. e., the possible lengths of the major semiaxes). All these solutions imply that the ellipsoid rotates about an axis immovable in space. They included all solutions known by that time — those obtained by Newton, Maclaurin, Jacobi, and Dedekind (for which the rotational axis coincides with one of the principal axes) and also new solutions (Riemann ellipsoids) for which the rotational axis lies in one of the principal planes of the ellipsoid.

3. Riemann used the energy integral of the system as the Lyapunov function (in modern terminology) to investigate the stability of shape-preserving motions (in the class of motions preserving the ellipsoidal shape); in this way, he found the Lyapunov-stability limits for the Maclaurin spheroids and Jacobi ellipsoids.

4. A particular case was noted in which a three-axial ellipsoid (unsteadily) rotates about one of the principal axes, and its semiaxes vary with time. This gives rise to a (Hamiltonian) system with two degrees of freedom for which Riemann noted an analogy with the motion of a material point on a two-dimensional surface of the form $xyz = \text{const}$ in a potential field of forces (it is this case that we will consider below in detail).

The study by Riemann was unique in terms of the importance of its results and possibilities of further generalizations; it was well in advance of its time.

There is also a study by Brioschi of 1861 [10], which was dedicated to lowering the order in the Dirichlet equations with the use of a decomposition into a potential and a vortical component. However, no substantial advance in the problem was associated with this work.

In his lectures in mechanics of 1876, Kirchhoff [11] also considered the motion of self-gravitating fluid ellipsoids. He noted that the d'Alembert principle is applicable to the Dirichlet motion (although he did not use it to derive the equation of motion). Kirchhoff presents a quadrature for the axisymmetric case and separately analyzes the case where the ellipsoid preserves the directions of its axes in space (a particular case of the motion considered by Riemann); Kirchhoff (following Riemann) conjectures that this problem also cannot be integrated in quadratures.

The possibility of applying the variational principle to the derivation of the equations of motion of a fluid ellipsoid was independently shown by Padova in 1871 [12] and Lipschitz in 1874 [13]. In the latter study [13], the problem of the motion of an elliptic cylinder was also formulated and integrated in quadratures.

Betti [14] also used the variational principle to derive the equations of motion of a fluid ellipsoid and represented these equations in a Lagrangian and a Hamiltonian form. However, as Tedone noted in his extensive survey [15], Betti made a mistake in his study when applying the variational principle to the derivation of the equation of motion of a homogeneous ellipsoid with an ellipsoidal fluid-density stratification. In this case, the hydrodynamic equations for the stratified, self-gravitating ellipsoid do not admit a solution with a linear dependence on the initial coordinates, which Betti considered (in view of the complex dependence of the gravitational potential inside the stratified ellipsoid). Nevertheless, all Betti's results remain valid for a constant density. Betti also represented the equations of motion in a Hamiltonian form (explicitly using the Poisson brackets on the $so(3)$ algebra) with a linear Poisson bracket and carried out a linear-integral-based reduction.

The above-listed results are the principal achievements of the classical period of the investigation of the dynamics of the Dirichlet ellipsoids.

General problems of the dynamics and statics of fluid ellipsoids, including the issues of stability, were investigated in classical treatises by Basset [16], Lamb [17], Thomson and Tait [18], Routh [19], in books by Appell [20], Lyttleton [21], in certain studies by Basset [22–24], Duhem [25], Hagen [26], Hicks [27], Hill [28], Love [29, 30], etc. Note also the following related subjects that constitute particular lines of research in this area.

• The investigation of figures of equilibrium bifurcating from the ellipsoid, e. g., pear-shaped figures, and the analysis of their stability (Lyapunov [31, 32], Poincaré [33, 34], Darwin [35], Jeans [4], and Sretenskiĭ [36]). As is known, a comprehensive analysis of this problem led Lyapunov to the development of a general theory of stability of motion, which goes under his name. Results concerning the figures of equilibrium were also obtained in classical studies by Giesen [37], Bryan [38], and Liouville [39]. Here, we do not touch upon the theory of stability, where many problems still remain open.

• Figures of equilibrium of a homogeneous fluid that is, however, stratified in a special manner. The theoretical analysis reached here its summit with a posthumous study by Lyapunov [32] (published by Stekloff), which still remains poorly comprehended. These investigations are closely related to the theory of the potential of stratified fluids (see Dyson [40], Ferrers [41], Volterra [42]). In [42], as in Lyapunov's studies on this subject (Lyapunov's results are presented from a more modern, functional-analysis standpoint un a book by Lichtenstein [43]), integral equations arise; they were later studied in the framework of functional analysis. In the context of this problem, let us mention a valuable but virtually forgotten study by Veronnet [44], a book by Pizzetti [45], and certain modern works [46–48] and [49] (the last publications do not contain any references to classical results and are highly controversial). A new period associated with astrophysical investigations was marked with investigations by Chandrasekhar and his school (see, e. g., [1, 50, 51]).

• In many problems that trace back to the classical Plateau experiments and the Bohr–Wheeler model of the atom [52], the surface-tension forces are considered instead of Newtonian attraction. For the comprehension of this question, we can recommend books by Appell [20] and Chandrasekhar [1]; it should be emphasized, however, that the theory presented there is much less advanced (see also [53, 55, 56], where a Pade approximation for the potential of the tension surface is given). At least, we are not aware of any results concerning the dynamics of fluid (masses) drops subjected to the action of surface-tension forces. Effects of viscosity on the dynamics in the Dirichlet–Riemann problem are discussed, e. g., in [57]. Early computer investigations of self-gravitating nonellipsoidal figures can be found, e. g., in [58].

• Note an interesting study by Narlikar and Larmor (1933) [59], where — likely, for the first time — the classical results (by Maclaurin, Jacobi, Dirichlet, Poincaré, etc.) are revised in the context of stellar-dynamical problems rather than planetary evolution. These investigators assumed that energy dissipation occurs in the process of stellar evolution. Later, this line of research was further developed by Chandrasekhar.

2. THE DIRICHLET AND RIEMANN EQUATIONS

2.1. The Dirichlet Equations

We recall here the principal steps in the derivation of the Dirichlet and Riemann equations and represent them in a modern matrix form.

The equations of the dynamics of a homogeneous, incompressible, ideal fluid of unit density in a Lagrangian form are in the case of potential forces applied to the fluid as follows:

$$\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^T \ddot{\boldsymbol{x}} = -\frac{\partial (U+p)}{\partial \boldsymbol{a}}, \tag{1}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$ are the initial positions of the material points of the medium (the so-called Lagrangian coordinates), $\boldsymbol{x}(\boldsymbol{a}, t)$ are the coordinates of the points of the medium at the time t (i. e., $\boldsymbol{x}(\boldsymbol{a}, 0) = a$), $U(\boldsymbol{a}, t)$ is the density of the potential energy of the external forces, $p(\boldsymbol{a}, t)$ is the pressure, and $\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} = \left\| \frac{\partial x_i}{\partial a_j} \right\|$ is the matrix of the partial derivatives. These equations must be supplemented with the incompressibility condition, which can be written is the case at hand as

$$\det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right) = 1. \tag{2}$$

Thus, we obtain a system of partial differential equations in which four quantities, viz., x_1, x_2, x_3, and p, are unknown as the functions of the variables $\boldsymbol{a}$ and t. To determine them, except initial conditions ($\boldsymbol{x}(\boldsymbol{a}, 0) = \boldsymbol{a}$, $\dot{\boldsymbol{x}}(\boldsymbol{a}, 0) = \boldsymbol{v}_0(\boldsymbol{a})$), also boundary conditions must be specified; in our case, the latter reduce to the statement that the pressure has the same value independent of $\boldsymbol{a}$ everywhere on the free surface.

Dirichlet noted that, if the potential of the external forces $U(\boldsymbol{a}, t)$ is a homogeneous quadratic function of the Lagrangian coordinates, i. e.

$$U(\boldsymbol{a}, t) = U_0(t) + (\boldsymbol{a}, \mathbf{V}(t)\boldsymbol{a}), \tag{3}$$

where $U_0(t)$ is independent of $\boldsymbol{a}$ and $\mathbf{V}(t)$ is a symmetric matrix, then the equations of motion (1), (2) admit a partial solution

$$\boldsymbol{x}(\boldsymbol{a}, t) = \mathbf{F}(t)\boldsymbol{a}, \quad \det \mathbf{F}(t) = 1. \tag{4}$$

Here, $\mathbf{F}(t)$ is a 3×3 matrix.

In this case, the boundary conditions will be satisfied provided that the fluid has initially an ellipsoidal shape,

$$(\boldsymbol{a}, \mathbf{A}_0^{-2}\boldsymbol{a}) \leqslant 1, \tag{5}$$

where $\mathbf{A}_0 = \mathrm{diag}(A_1^0, A_2^0, A_3^0)$ is the matrix of the initial semiaxes and the pressure has the form

$$p(\boldsymbol{a}, t) = p_0(t) + \sigma(t)(1 - (\boldsymbol{a}, \mathbf{A}_0^{-2}\boldsymbol{a})). \tag{6}$$

We substitute (3), (4), and (6) into (1) and (2) to obtain equations for the matrix $\mathbf{F}(t)$ and the function $\sigma(t)$ in the form

$$\begin{aligned} &\mathbf{F}^T\ddot{\mathbf{F}} = -2\mathbf{V} - 2\sigma\mathbf{A}_0^{-2}, \\ &\det\mathbf{F} = 1. \end{aligned} \qquad (\textit{the Dirichlet equations}) \tag{7}$$

As Dirichlet showed, the system of ten equations (7) for ten unknown functions $F_{ij}(t), \sigma(t)$, $i, j = 1, 2, 3$, is compatible.

Obviously, the transformation (4) changes the original ellipsoid (5) into the ellipsoid specified by the quadratic form

$$(\boldsymbol{x}, (\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T)^{-1}\boldsymbol{x}) \leqslant 1. \tag{8}$$

2.2. The Riemann Equations

Before writing an explicit expression for the potential (3) and, accordingly, the right-hand side of equations (7), let us show how the equations of motion can be written in a Riemannian form. To this end, we pass to the moving system of the principal axes of the ellipsoid. It is known that such a transformation is given by the orthogonal matrix

$$\boldsymbol{\zeta} = \mathbf{Q}\mathbf{x}, \quad \mathbf{Q}^T = \mathbf{Q}^{-1}. \tag{9}$$

In the new coordinates $\boldsymbol{\zeta}$, the ellipsoid is specified by the relationship

$$(\boldsymbol{\zeta}, \mathbf{A}^{-2}\boldsymbol{\zeta}) \leqslant 1, \tag{10}$$

where $\mathbf{A} = \mathrm{diag}(A_1, A_2, A_3)$ is the matrix of the principal semiaxes at the given time.

We also note that, since the transform (4) is linear, the fluid particles constantly move over ellipsoids for which

$$(\boldsymbol{\zeta}, \mathbf{A}^{-2}\boldsymbol{\zeta}) = (\boldsymbol{a}, \mathbf{A}_0^{-2}\boldsymbol{a}) = n^2 = \mathrm{const}, \quad 0 \leqslant n^2 < 1. \tag{11}$$

(In particular, the fluid particles that were initially at the boundary remain at the boundary at any time). Therefore, the modulus of the vector $\mathbf{A}^{-1}\boldsymbol{\zeta}$ does not vary, so that the vectors $\mathbf{A}^{-1}\boldsymbol{\zeta}$ and $\mathbf{A}_0^{-1}\boldsymbol{a}$ are also related by the orthogonal transformation

$$\mathbf{A}^{-1}\boldsymbol{\zeta} = \boldsymbol{\Theta}\mathbf{A}_0^{-1}\boldsymbol{a}, \quad \boldsymbol{\Theta}^T = \boldsymbol{\Theta}^{-1}. \tag{12}$$

Thus, we obtain the following decomposition of the matrix $\mathbf{F}$:

$$\mathbf{F} = \mathbf{Q}^T\mathbf{A}\boldsymbol{\Theta}\mathbf{A}_0^{-1}. \tag{13}$$

Remark 1. Multiplying by a constant matrix $\mathbf{A}_0$ yields a decomposition of the form

$$\mathbf{F}\mathbf{A}_0 = \mathbf{Q}^T\mathbf{A}\boldsymbol{\Theta} \tag{14}$$

known in linear algebra as a singular decomposition [60].

We introduce the angular velocities corresponding to the orthogonal transformations,

$$\mathbf{w} = \dot{\mathbf{Q}}\mathbf{Q}^T, \quad \boldsymbol{\omega} = \dot{\boldsymbol{\Theta}}\boldsymbol{\Theta}^T, \tag{15}$$

which are known to be antisymmetric matrices [63]. The substitution of (15) into equations (7) yields the Riemann equations, which can be written in the following matrix form:

$$\begin{aligned} &\dot{\mathbf{v}} - \mathbf{w}\mathbf{v} + \mathbf{v}\boldsymbol{\omega} = -2\hat{\mathbf{V}}\mathbf{A} + 2\sigma\mathbf{A}^{-1}, \\ &\mathbf{v} = \dot{\mathbf{A}} - \mathbf{w}\mathbf{A} + \mathbf{A}\boldsymbol{\omega}, \qquad (\textit{the Riemann equations}) \\ &A_1A_2A_3 = 1, \end{aligned} \tag{16}$$

where $\hat{\mathbf{V}} = \mathbf{A}^{-1}\boldsymbol{\Theta}\mathbf{A}_0\mathbf{V}\mathbf{A}_0\boldsymbol{\Theta}^T\mathbf{A}^{-1}$.

We complement this system with the equations of evolution of the matrices $\mathbf{Q}$ and $\boldsymbol{\Theta}$,

$$\dot{\mathbf{Q}} = \mathbf{w}\mathbf{Q}, \quad \dot{\boldsymbol{\Theta}} = \boldsymbol{\omega}\boldsymbol{\Theta}, \tag{17}$$

to obtain the complete system of equations of motion describing the dynamics of the fluid ellipsoid.

Remark 2. For an arbitrary matrix $\mathbf{G} \in GL(3)$ with differing eigenvalues, the decomposition $\mathbf{G} = \mathbf{Q}\mathbf{A}\boldsymbol{\Theta}$ with $\mathbf{Q}, \boldsymbol{\Theta} \in SO(3)$, $\mathbf{A} = \operatorname{diag}(a_1, a_2, a_3)$, $a_1 > a_2 > a_3$ is not unique. Indeed, the matrices $\mathbf{Q}$ and $\boldsymbol{\Theta}$ admit discrete transforms of the form [60]

$$\begin{gathered} \mathbf{Q}' = \mathbf{Q}\mathbf{R}_i, \quad \boldsymbol{\Theta}' = \boldsymbol{\Theta}\mathbf{R}_i, \quad i = 0, 1, 2, 3, \\ \mathbf{R}_0 = \mathbf{E}, \quad \mathbf{R}_1 = \operatorname{diag}(1, -1, -1), \quad \mathbf{R}_2 = \operatorname{diag}(-1, 1, -1), \quad \mathbf{R}_3 = \operatorname{diag}(-1, -1, 1). \end{gathered} \tag{18}$$

Therefore, the space $\mathbb{R}^2 \otimes SO(3) \otimes SO(3)$ (which is diffeomorphous to the configuration space of the Riemann system) is a four-sheet covering of the real configuration space $SL(3)$. A similar procedure is used for a quaternion representation of the equations of a rigid body [63].

2.3. Gravitational potential

Now, we determine the right-hand sides of equations (7) and (17). We use the known representation of the gravitational potential for the interior of the ellipsoid in the system of the principal axes

$$U(\boldsymbol{\zeta}) = -\frac{3}{4}mG\int_0^\infty \frac{d\lambda}{\Delta(\lambda)}\left(1 - \sum_i \frac{\zeta_i^2}{A_i^2 + \lambda}\right), \quad \Delta^2(\lambda) = \prod_i (A_i^2 + \lambda), \tag{19}$$

where G is the gravitational constant and $m = \frac{4}{3}\pi\rho A_1A_2A_3$ is the mass of the ellipsoid.

It is now necessary to represent (19) in terms of the elements of the transformation matrix $\mathbf{F}$ and in the Lagrangian coordinates $\boldsymbol{a}$. We use (13) to find $\mathbf{A} = \mathbf{Q}\mathbf{F}\mathbf{A}_0\boldsymbol{\Theta}^T$ and obtain

$$\begin{aligned} &\mathbf{A}^2 = \mathbf{A}\mathbf{A}^T = \mathbf{Q}\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T\mathbf{Q}^T, \\ &\Delta^2(\lambda) = \det(\mathbf{A}^2 + \lambda\mathbf{E}) = \det(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T + \lambda\mathbf{E}), \\ &\sum_i \frac{\zeta_i^2}{A_i^2 + \lambda} = (\boldsymbol{\zeta}, (\mathbf{A}^2 + \lambda\mathbf{E})^{-1}\boldsymbol{\zeta}) = (\boldsymbol{a}, \mathbf{F}^T(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T + \lambda\mathbf{E})^{-1}\mathbf{F}\boldsymbol{a}). \end{aligned} \tag{20}$$

Thus, we find the following representation for the matrix $\mathbf{V}$ in the Dirichlet equations:

$$\mathbf{V} = \varepsilon\int_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T + \lambda\mathbf{E})}}\mathbf{F}^T(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T + \lambda\mathbf{E})^{-1}\mathbf{F}, \quad \varepsilon = \frac{3}{4}mG; \tag{21}$$

it can be shown by direct calculations (see [2]) that $\mathbf{V}$ depends on the elements of the matrix $\mathbf{F}$ only through symmetric combinations of the form $\Phi_{ij} = \sum_k F_{ik}F_{jk}$, which are the dot products of columns of the matrix $\mathbf{F}$.

The relationship $\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\zeta}} = \mathbf{A}_0\boldsymbol{\Theta}^T\mathbf{A}^{-1}$ can be used to easily show that, in the Riemann equations, $\hat{\mathbf{V}} = \operatorname{diag}(\hat{V}_1, \hat{V}_2, \hat{V}_3)$, where

$$\hat{V}_i = \varepsilon\int_0^\infty \frac{1}{\lambda + A_i^2}\frac{d\lambda}{\Delta(\lambda)} = -\frac{1}{A_i}\frac{\partial}{\partial A_i}\varepsilon\int_0^\infty \frac{d\lambda}{\Delta(\lambda)}. \tag{22}$$

2.4. The Roche Problem

By the Roche problem, according to Jeans's terminology [4] (see also [1, 35]) we mean the problem of the interaction of a deformable body (satellite) and a spherical rigid body which move along circular Keplerian orbits. Actually, in [64] Roche considered the motion of the liquid mass under the action of a gravitating center (the notion of *Roche zones* traces back to this work). More general problem, where the second body does not have a spherical symmetry (i.e. the motion of two arbitrary bodies with mass centers moving along circular orbits), is called the Darwin problem [35].

Let a self-gravitating fluid mass move in the field of a spherically symmetric rigid body and both of these bodies rotate about their common center of mass in circular orbits. We choose a (moving) coordinate system $Ox_1x_2x_3$ with its origin at the center of mass of the ellipsoid and direct the Ox_1 axis toward the common center and the Ox_3 axis normally to the plane of rotation (see Fig. 1).

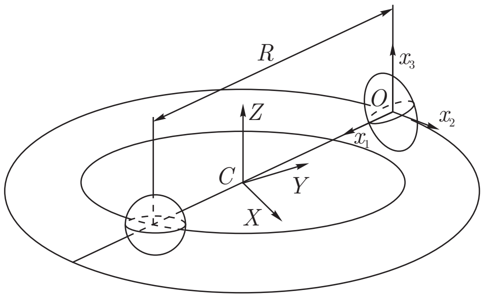

Fig. 1.

The equations of motion of incompressible fluid can be written in this case in the following Lagrangian form

$$\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^T (\ddot{\boldsymbol{x}} + 2\omega \boldsymbol{e}_3 \times \dot{\boldsymbol{x}}) = -\frac{\partial}{\partial \boldsymbol{a}}\left(p + U + U_s - \frac{1}{2}\omega^2(x_1^2 + x_2^2) + \omega^2 \frac{m_s}{m_e + m_s} R x_1\right), \tag{23}$$

$$\det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right) = 1, \tag{24}$$

where, as before, $\boldsymbol{a}$ are the Lagrangian coordinates of fluid elements, $\boldsymbol{x}(\boldsymbol{a}, t)$ are their positions at the given time, $p(\boldsymbol{a}, t)$ is the pressure, R is the distance between the centers of mass of the bodies, m_e and m_s are the masses of the ellipsoid and the sphere, respectively, ω is the angular velocity of rotation of the system about their common center of mass, and U is the gravitational potential (19). The gravitational potential of a spherical body U_s has the form

$$U_s = -\frac{m_s G}{\sqrt{(x_1 - R)^2 + x_2^2 + x_3^2}} = -\frac{m_s G}{R}\left(1 + \frac{x_1}{R} + \frac{1}{2}\frac{1}{R^2}\left(2x_1^2 - x_2^2 - x_3^2\right) + \ldots\right),$$

where G is the gravitational constant.

We omit higher-order terms in $\frac{|\boldsymbol{x}|}{R}$ and use the well-known relationship for a circular Keplerian orbit $R^3\omega^2 = G(m_e + m_s)$ to obtain finally (after collecting like terms) the equation

$$\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^T (\ddot{\boldsymbol{x}} + 2\omega \boldsymbol{e}_3 \times \dot{\boldsymbol{x}}) = -\frac{\partial}{\partial \boldsymbol{a}}\left(p + U - \frac{1}{2}\omega^2(\boldsymbol{x}, \mathbf{B}\boldsymbol{x})\right), \ \det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right) = 1, \tag{25}$$

where $\mathbf{B} = \operatorname{diag}\left(\frac{3m_s + m_e}{m_e + m_s}, \frac{m_e}{m_e + m_s}, -\frac{m_s}{m_e + m_s}\right)$. In the limiting case of a motionless Newtonian center $\left(\frac{m_e}{m_s} \to 0\right)$, we have $\mathbf{B} = \operatorname{diag}(3, 0, -1)$.

By substituting (4) into (6), we obtain the equations of motion in Roche's problem in the form

$$\mathbf{F}^T(\ddot{\mathbf{F}} + 2\boldsymbol{\Omega}\dot{\mathbf{F}}) = -2\mathbf{V} + 2\sigma\mathbf{A}_0^{-2} + \omega^2\mathbf{F}^T\mathbf{B}\mathbf{F},$$
$$\det\mathbf{F} = 1, \tag{26}$$

where $\boldsymbol{\Omega} = \| - \omega\varepsilon_{ijk}\|$ is the matrix of the rotational velocity.

Remark 3. Equations (25) are given in the book by Chandrasekhar, who uses them only to find hydrostatically equilibrated configurations of fluid masses and analyze their stability. Chandrasekhar does not present the dynamical equations (26).

Obviously, equations (26) can also be written in the Riemann form as

$$\begin{aligned} &\dot{\mathbf{v}} - \mathbf{w}\mathbf{v} - \mathbf{v}\mathbf{w} + 2\bar{\boldsymbol{\Omega}}\mathbf{v} = -2\hat{\mathbf{V}}\mathbf{A} + 2\sigma\mathbf{A}^{-1} + \omega^2\mathbf{A}\bar{\mathbf{B}}\mathbf{A}, \\ &\mathbf{v} = \dot{\mathbf{A}} - \mathbf{w}\mathbf{A} + \mathbf{A}\mathbf{w}, \\ &A_1A_2A_3 = 1, \end{aligned} \tag{27}$$

where $\bar{\boldsymbol{\Omega}} = \mathbf{Q}\boldsymbol{\Omega}\mathbf{Q}^T and \bar{\mathbf{B}} = \mathbf{Q}\mathbf{B}\mathbf{Q}^T$ are the matrices reduced to the principal axes of the ellipsoid. As before, equations (17) should be added.

Note that equations (27) in this case do not form a closed system (in contrast to the Riemann equations), and the system closes only upon adding equations (17) for the evolution of the matrix $\mathbf{Q}$.

3. FIRST INTEGRALS

Let us return to the Dirichlet–Riemann problem on dynamics of the self-gravitating ellipsoid.

The first integrals of the equations, linear in the velocities, can be obtained from the conservation laws for vorticity and angular momentum (the law of areas).

3.1. Vorticity

We write the law of conservation of vorticity for the hydrodynamic equations in the Lagrangian form (1), thus obtaining

$$\sum_i \left(\frac{\partial x_i}{\partial a_k}\frac{\partial \dot{x}_i}{\partial a_l} - \frac{\partial x_i}{\partial a_l}\frac{\partial \dot{x}_i}{\partial a_k} \right) = \xi_{kl} = \text{const}, \tag{28}$$

with the condition $\xi_{kl} = -\xi_{lk}$ satisfied. We denote this antisymmetric matrix as $\boldsymbol{\Xi} = \|\xi_{kl}\|$ and find for the Dirichlet equations (7) that

$$\boldsymbol{\Xi} = \mathbf{F}^T\dot{\mathbf{F}} - \dot{\mathbf{F}}^T\mathbf{F} = \text{const}. \tag{29}$$

In the Riemannian variables, we obtain

$$\boldsymbol{\Xi}' = \mathbf{A}_0\boldsymbol{\Xi}\mathbf{A}_0 = \boldsymbol{\Theta}^T(\mathbf{A}^2\boldsymbol{\omega} + \boldsymbol{\omega}\mathbf{A}^2 - 2\mathbf{A}\mathbf{w}\mathbf{A})\boldsymbol{\Theta} = \text{const}. \tag{30}$$

A straightforward proof of the conservation of vorticity $\boldsymbol{\Xi}$ based on the Dirichlet equations (7) is obvious (since the right-hand side is a symmetric matrix).

As already mentioned, the conservation of vorticity in this problem was noted by Dirichlet even before the appearance of a classical study by Helmholtz in which this law was extended to ideal hydrodynamics on the whole.

3.2. Momentum

The angular momentum relative to the center of the ellipsoid can be represented as

$$M_{ij} = \int (x_i \dot{x}_j - x_j \dot{x}_i) d^3\boldsymbol{x} = \frac{m}{5} \sum_k (F_{ik}\dot{F}_{jk} - F_{jk}\dot{F}_{ik})(A_k^0)^2. \tag{31}$$

In a matrix form, with the unimportant multiplier omitted, we have

$$\mathbf{M}' = \mathbf{F}\mathbf{A}_0^2\dot{\mathbf{F}}^T - \dot{\mathbf{F}}\mathbf{A}_0^2\mathbf{F}^T = \mathrm{const}, \tag{32}$$

where $\mathbf{M}' = \|\frac{5}{m} M_{ij}\|$. Similarly, in the Riemannian variables, we have

$$\mathbf{M}' = \mathbf{Q}^T(\mathbf{A}^2\mathbf{w} + \mathbf{w}\mathbf{A}^2 - 2\mathbf{A}\boldsymbol{\omega}\mathbf{A})\mathbf{Q} = \mathrm{const}. \tag{33}$$

The use of the Riemann equations (16) is convenient in proving the invariability of the momentum $\mathbf{M}$ (the relevant calculations are also straightforward in this case).

3.3. Energy

In addition to the linear integrals, the equations of motion also admit another, quadratic integral, viz., the total energy of the system. The integration of the kinetic and the potential energy of the fluid particles over the volume of the ellipsoid yields

$$\begin{gathered}\mathcal{E} = \frac{m}{5}(T_e + U_e), \\ T_e = \frac{1}{2}\,\mathrm{Tr}(\dot{\mathbf{F}}\mathbf{A}_0^2\dot{\mathbf{F}}^T) = \frac{1}{2}\,\mathrm{Tr}(\dot{\mathbf{A}}^2 - \mathbf{w}^2\mathbf{A}^2 - \boldsymbol{\omega}^2\mathbf{A}^2 + 2\mathbf{A}\mathbf{w}\mathbf{A}\boldsymbol{\omega}), \\ U_e = -2\varepsilon \int_0^\infty \frac{d\lambda}{\sqrt{(\lambda + A_1^2)(\lambda + A_2^2)(\lambda + A_3^2)}}. \end{gathered} \tag{34}$$

4. LAGRANGIAN AND HAMILTONIAN FORMALISM

4.1. Hamiltonian Principle and Lagrangian Formalism

It is known (see, e. g., [11]) that the motion of ideal fluid satisfies the Hamilton principle; therefore, Dirichlet's solution also satisfies this principle. This makes it possible to represent the equations of motion in a Lagrangian and, next, in a Hamiltonian form. The Hamiltonian principle for the considered problem was used for the first time by Lipschitz [13] and Padova [12].

As the Lagrangian function, it is necessary to choose the difference between the kinetic and potential energies of the fluid in the ellipsoid; within the unimportant multiplier, we have

$$L = T_e - U_e, \tag{35}$$

where T_e and U_e were defined above in (34). The elements of the matrix $\mathbf{F}$ appear as generalized coordinates. We write the Lagrange–Euler equations taking into account the constraint $\det\mathbf{F} = 1$ to obtain

$$\left(\frac{\partial L}{\partial \dot{\mathbf{F}}}\right)^{\cdot} - \frac{\partial L}{\partial \mathbf{F}} = \kappa \frac{\partial \varphi}{\partial \mathbf{F}}, \tag{36}$$

where $\varphi = \det\mathbf{F}$, and use the following matrix notation for any function: $\frac{\partial f}{\partial \mathbf{F}} = \left\|\frac{\partial f}{\partial F_{ij}}\right\|$, $\frac{\partial f}{\partial \dot{\mathbf{F}}} = \left\|\frac{\partial f}{\partial \dot{F}_{ij}}\right\|$, κ being the undefined Lagrangian multiplier. The differentiation in view of the formula $\left(\frac{\partial\varphi}{\partial\mathbf{F}}\right)^T = \varphi\mathbf{F}^{-1}$ yields

$$\ddot{\mathbf{F}}\mathbf{A}_0^2 = 2\varepsilon\frac{\partial}{\partial\mathbf{F}}\int_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T + \lambda\mathbf{E})}} + \kappa(\mathbf{F}^{-1})^T\det\mathbf{F}. \tag{37}$$

We can easily make sure that these equations coincide with the Dirichlet equations (7) if we set $\kappa = 2\sigma$.

The matrix of the initial semiaxes $\mathbf{A}_0$ appears in the Lagrangian function and the equations of motion of the system as a set of parameters. Obviously, these parameters can be transferred to the initial conditions; indeed, upon the substitution $\mathbf{G} = \mathbf{F}\mathbf{A}_0$ (suggested by Dedekind [9]), the Lagrangian function and the equation of constraint can be written as

$$L = \frac{1}{2}\operatorname{Tr}(\dot{\mathbf{G}}\dot{\mathbf{G}}^T) + 2\varepsilon \int\limits_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{G}\mathbf{G}^T + \lambda\mathbf{E})}}, \qquad \varphi = \det\mathbf{G} = \det\mathbf{A}_0 = \text{const}. \tag{38}$$

The initial conditions have obviously the form $\mathbf{G}|_{t=0} = \mathbf{A}_0$, and the equation of motion preserves its form, $\left(\frac{\partial L}{\partial \dot{\mathbf{G}}}\right)^{\cdot} - \frac{\partial L}{\partial \mathbf{G}} = \widetilde{\mathcal{H}}\frac{\partial \varphi}{\partial G}$.

It can also be shown that the substitution

$$\mathbf{G} \to (\det\mathbf{A}_0)^{1/3}\mathbf{G}, \quad t \to \frac{(\det\mathbf{A}_0)^{1/3}}{2\varepsilon}t$$

reduces the system (38) to the case of $\varepsilon = 1/2, \varphi = 1$. Thus, the dynamics of the self-gravitating fluid ellipsoid is described by a natural Lagrangian system without parameters on the $SL(3)$ group.

The first integrals — vorticity (30), momentum (32), and energy (34) — can be represented in the form

$$\mathbf{\Xi} = \mathbf{G}^T\dot{\mathbf{G}} - \dot{\mathbf{G}}^T\mathbf{G}, \quad \mathbf{M} = \mathbf{G}\dot{\mathbf{G}}^T - \dot{\mathbf{G}}\mathbf{G}^T, \qquad \mathcal{E} = \frac{1}{2}\operatorname{Tr}(\dot{\mathbf{G}}\dot{\mathbf{G}}^T) - 2\varepsilon \int\limits_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{G}\mathbf{G}^T + \lambda\mathbf{E})}}. \tag{39}$$

Riemann used the decomposition (13) to represent the equations of motion on the configuration space $\mathbb{R}^2 \otimes \mathrm{SO}(3) \otimes \mathrm{SO}(3)$ (the direct product of the Abel group of translations and two copies of the group of rotations of three-dimensional space), with the elements of the matrices $\mathbf{w}$ and $\boldsymbol{\omega}$ corresponding to the velocity components with respect to the basis of left-invariant vector fields. The equations of motion assume the form of the Poincaré equations on the Lie group [63]; in view of the fact that the Lagrangian function (38) is independent of the elements of the matrices $\mathbf{Q}$ and $\mathbf{\Theta}$ and with due account for the constraint $\varphi = A_1A_2A_3 = \text{const}$, we obtain the following representation of the Riemann equations:

$$\left(\frac{\partial L}{\partial \dot{A}_i}\right)^{\cdot} = \frac{\partial L}{\partial A_i} + \tilde{\kappa}\frac{\partial \varphi}{\partial A_i}, \qquad \left(\frac{\partial L}{\partial \mathrm{w}_i}\right)^{\cdot} = \sum_{j,k}\varepsilon_{ijk}\frac{\partial L}{\partial \mathrm{w}_j}\mathrm{w}_k, \quad \left(\frac{\partial L}{\partial \omega_i}\right)^{\cdot} = \sum_{j,k}\varepsilon_{ijk}\frac{\partial L}{\partial \omega_j}\omega_k. \tag{40}$$

where $\tilde{\kappa}$ is the Lagrangian undetermined multiplier (which coincides with σ within a multiplier) and ε_{ijk} is the Levi-Civita antisymmetric tensor.

From here on, the components w_i and ω_i are related to the elements of the antisymmetric matrices (15) according to the regular rule

$$\mathrm{w}_{ij} = \varepsilon_{ijk}\mathrm{w}_k, \quad \omega_{ij} = \varepsilon_{ijk}\omega_k. \tag{41}$$

4.2. Symmetry Group and the Dedekind Reciprocity Law

The Lagrangian representation of the Dirichlet equations (36) offers a very simple way to finding the symmetry group of the system. Indeed, it can be shown that the Lagrangian with the constraint [see (38)] and, therefore, the equations of motion are invariant with respect to transformations of the form

$$\mathbf{G}' = \mathbf{S}_1\mathbf{G}\mathbf{S}_2, \quad \mathbf{S}_1, \mathbf{S}_2 \in SO(3). \tag{42}$$

Thus, the system is invariant with respect to the group $\Gamma = SO(3) \otimes SO(3)$.

Clearly, the Noether integrals corresponding to the transformations (42) are the integrals of vorticity and total momentum (39). Accordingly, as will be shown below, the Riemann equations describe a system reduced based on the given symmetry group.

Furthermore, it can easily be shown using (38) that the equations of motion are invariant with respect to the discrete transformation of transposition of matrices:

$$\mathbf{G}' = \mathbf{G}^T.$$

Therefore, we have

Theorem 1 (The Dedekind reciprocity law). *Any solution,* $\mathbf{G}(t)$*, of the Dirichlet equations can be placed in correspondence with the solution* $\mathbf{G}'(t) = \mathbf{G}^T(t)$ *for which the rotation of the ellipsoid and the rotation of the fluid inside the ellipsoid (i. e.,* $\mathbf{\Theta}$ *and* $\mathbf{Q}$*; see (13)) are interchanged.*

The most widely known example is the Dedekind ellipsoid reciprocal to the Jacobi ellipsoid. In this case, the axes of the three-axial ellipsoid are spatially invariable and the fluid inside it moves around the minor axis in closed ellipses [3, 9].

4.3. Hamiltonian Formalism and Symmetry-based Reduction

We represent the Riemann equations in a Hamiltonian form. To this end, we first use the constraint equation $\varphi = \text{const}$ to find a representation of one semiaxis,

$$A_3 = \frac{v_0}{A_1A_2}, \tag{43}$$

where v_0 is the volume of the ellipsoid (within a multiplier). We carry out the Legendre transformation

$$\begin{gathered} p_i = \frac{\partial L}{\partial \dot{A}_i}, \quad m_k = \frac{\partial L}{\partial \mathrm{w}_k}, \quad \mu_k = \frac{\partial L}{\partial \omega_k}, \quad i = 1, 2, \quad k = 1, 2, 3, \\ H = \sum_i p_i\dot{A}_i + \sum_k (m_k\mathrm{w}_k + \mu_k\omega_k) - L\,|_{\dot{A},\omega,\mathrm{w}\to p,m,\mu}\,. \end{gathered} \tag{44}$$

It can be shown using the expressions for the integrals, (30) and (33), that the vectors $\boldsymbol{m} = (m_1, m_2, m_3)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ are related to the momentum and vorticity of the ellipsoid via the formulas

$$\boldsymbol{m} = \mathbf{Q}^T\boldsymbol{M}', \qquad \boldsymbol{\mu} = \mathbf{\Theta}^T\boldsymbol{\xi}', \tag{45}$$

where the vectors $\boldsymbol{M}'$ and $\boldsymbol{\xi}'$ are constituted by the components of the antisymmetric matrices $\mathbf{M}'$ and $\mathbf{\Xi}'$ according to the normal rule (41). In the new variables, the equations of motion assume the form

$$\begin{gathered} \dot{A}_i = \frac{\partial H}{\partial \dot{p}_i}, \quad \dot{p}_i = \frac{\partial H}{\partial A_i}, \quad i = 1, 2, \\ \dot{\boldsymbol{m}} = \boldsymbol{m} \times \frac{\partial H}{\partial \boldsymbol{m}}, \quad \dot{\boldsymbol{\mu}} = \boldsymbol{\mu} \times \frac{\partial H}{\partial \boldsymbol{\mu}}. \end{gathered} \tag{46}$$

Here, the Hamiltonian is

$$H = H_A + H_{m\mu} + U_e,$$

$$H_A = \frac{1}{2}\frac{A_3^{-2}(p_1^2+p_2^2)+(p_1A_2^{-1}-p_2A_1^{-1})^2}{\sum A_i^{-2}}, \tag{47}$$
$$H_{m\mu} = \frac{1}{4}\sum_{\text{cycle}}\left(\frac{m_i+\mu_i}{A_j-A_k}\right)^2+\left(\frac{m_i-\mu_i}{A_j+A_k}\right)^2,$$

where U_e is specified by formula (34) and it is assumed that A_3 is defined according to (43).

In addition, equations (46) must necessarily be supplemented with equations describing the evolution of the matrices $\mathbf{Q}$ and $\mathbf{\Theta}$; they have the form

$$\dot{Q}_{ij} = \sum_{k,l}\varepsilon_{ikl}Q_{kj}\frac{\partial H}{\partial m_l}, \quad \dot{\Theta}_{ij} = \sum_{k,l}\varepsilon_{ikl}\Theta_{kj}\frac{\partial H}{\partial \mu_l}. \tag{48}$$

Equations (46) and (48) form a Hamiltonian system with eight degrees of freedom and uncanonical Poisson brackets,

$$\{A_i,p_j\}=\delta_{ij}, \quad \{m_i,m_j\}=\varepsilon_{ijk}m_k, \quad \{\mu_i,\mu_j\}=\varepsilon_{ijk}\mu_k, \tag{49}$$

$$\{m_r,Q_{jk}\}=\varepsilon_{ikl}Q_{jl}, \quad \{\mu_i,\Theta_{jk}\}=\varepsilon_{ikl}\Theta_{jl}, \tag{50}$$

where zero brackets are omitted.

Remark 4. The elimination of one semiaxis (43) results in the loss of symmetry of the Hamiltonian (47); therefore, the equations for the semiaxes A_i are normally left in the Lagrangian form with an undetermined multiplier [1, 3].

It can be seen from the above relationships that the system of equations (46), which describes the evolution of the variables $A_i, p_i, \boldsymbol{m}$, and $\boldsymbol{\mu}$, separates; in addition, the Poisson bracket of these variables, (49), also proves to be closed. It is not difficult to show that that equations (46) describe a system reduced over the symmetry group (42).

Limitation: the brackets (49) obviously have two Casimir functions,

$$\Phi_m = (\boldsymbol{m},\boldsymbol{m}), \quad \Phi_\mu = (\boldsymbol{\mu},\boldsymbol{\mu}), \tag{51}$$

and have a rank of eight (provided that $\Phi_m \neq 0, \Phi_\mu \neq 0$).

Therefore, the reduced system has generally four degrees of freedom.

In particular cases where *one of the integrals* (51) *is zero, the reduced system has three degrees of freedom.* These are so-called irrotational ($\Phi_\mu = 0$) and momentum-free ($\Phi_m = 0$) ellipsoids.

If both of the integrals (51) *vanish, the reduced system has two degrees of freedom* and describes oscillations of the ellipsoid without changes in the directions of the axes and without inner flows (this case will be considered below in detail).

Remark 5. The canonical variables in the Riemann equations were introduced for the first time by Betti [14], who used the commutation representations of the $so(4)$ algebra long before the advent of the modern theory of Hamiltonian systems on the Lie algebras. With the use of commutation, he introduced, in a quite modern way, canonical variables to reduce the integration of the Riemann equations to the integration of the Hamilton–Jacobi equations. The Hamiltonian nature of the Riemann equations is also considered in modern studies [53–55], which are related to the representation of the equations of motion on an extended Lie algebra for which the actual motions are in special orbits; the value of such a calculation for dynamics is not yet clear to us. A more formal procedure of reduction and Hamiltonization of the Riemann equations nearly relevant to our study is described in [60]. An akin analysis is done in [61] in the context of the Dirichlet motions in ideal magnetohydrodynamics. An alternative approach to the Hamiltonian nature, which also should be discussed, is presented in [62].

Remark 6. The linear transformation

$$\boldsymbol{L} = \boldsymbol{m} + \boldsymbol{\mu}, \quad \boldsymbol{\pi} = \boldsymbol{m} - \boldsymbol{\mu}$$

reduces the angular part $H_{m\mu}$ of the Hamiltonian (47) to a diagonal form

$$\begin{gathered} H_{m\mu} = \frac{1}{4}(\boldsymbol{L}, \boldsymbol{\Lambda}\boldsymbol{L}) + \frac{1}{4}(\boldsymbol{\pi}, \boldsymbol{\Pi}\boldsymbol{\pi}), \\ \boldsymbol{\Lambda} = \operatorname{diag}\left(\frac{1}{(A_2 - A_3)^2}, \frac{1}{(A_3 - A_1)^2}, \frac{1}{(A_1 - A_2)^2}\right), \\ \boldsymbol{\Pi} = \operatorname{diag}\left(\frac{1}{(A_2 + A_3)^2}, \frac{1}{(A_3 + A_1)^2}, \frac{1}{(A_1 + A_2)^2}\right); \end{gathered} \tag{52}$$

in this case, the Poisson brackets reduce to the form

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k, \quad \{L_i, \pi_j\} = \varepsilon_{ijk}\pi_k, \quad \{\pi_i, \pi_j\} = \varepsilon_{ijk} L_k$$

and correspond, as is known, to an $so(4)$ algebra. The corresponding equations of motion can be represented in the matrix form

$$\dot{\mathbf{X}} = [\mathbf{X}, \mathbf{\Omega}], \tag{53}$$

where

$$\mathbf{X} = \begin{Vmatrix} 0 & L_3 & -L_2 & \pi_1 \\ -L_3 & 0 & L_1 & \pi_2 \\ L_2 & -L_1 & 0 & \pi_3 \\ -\pi_1 & -\pi_2 & -\pi_3 & 0 \end{Vmatrix}, \quad \mathbf{\Omega} = \begin{Vmatrix} 0 & \frac{\partial H}{\partial L_3} & -\frac{\partial H}{\partial L_2} & \frac{\partial H}{\partial \pi_1} \\ -\frac{\partial H}{\partial L_3} & 0 & \frac{\partial H}{\partial L_1} & \frac{\partial H}{\partial \pi_2} \\ \frac{\partial H}{\partial L_2} & -\frac{\partial H}{\partial L_1} & 0 & \frac{\partial H}{\partial \pi_3} \\ -\frac{\partial H}{\partial \pi_1} & -\frac{\partial H}{\partial \pi_2} & -\frac{\partial H}{\partial \pi_3} & 0 \end{Vmatrix}, \tag{54}$$

and the equations for A_i and p_i preserve their previous form (46).

Equations (53) coincide in their form with the equations of motion of a free four-dimensional rigid body. In the dynamics of the rigid body, the momentum and angular velocity are in this case linked by linear relationships of the form

$$\mathbf{X} = \frac{1}{2}(\mathbf{J}\mathbf{\Omega} + \mathbf{\Omega}\mathbf{J}), \tag{55}$$

where the constant symmetrical matrix $\mathbf{J}$ is the moment of inertia of the body with respect to axes fixed to the body. It can easily be shown that the matrices (54) for the Dirichlet–Riemann problem do not satisfy the relationship (55) at any matrix $\mathbf{J}$, i. e., the analogy with the dynamics of the rigid body is purely formal in this case. Recall that the matrices $\boldsymbol{\Lambda}$ and $\boldsymbol{\Pi}$ for a four-dimensional rigid body have the form [63]

$$\boldsymbol{\Lambda} = \operatorname{diag}\left(\frac{1}{\lambda_2 + \lambda_3}, \frac{1}{\lambda_1 + \lambda_3}, \frac{1}{\lambda_1 + \lambda_2}\right), \quad \boldsymbol{\Pi} = \operatorname{diag}\left(\frac{1}{\lambda_0 + \lambda_1}, \frac{1}{\lambda_0 + \lambda_2}, \frac{1}{\lambda_0 + \lambda_3}\right).$$

This form of equations (53) was noted by Dyson [77] for the case of the dynamics of a compressible ellipsoid (see below).

5. PARTICULAR CASES OF MOTION

5.1. Shape-preserving Motions of the Ellipsoin

The simplest motions of the fluid ellipsoids are represented by a family of solutions for which all the three axes of the ellipsoid are time-independent,

$$A_i = \text{const}, \quad i = 1, 2, 3. \tag{56}$$

Clearly, the Maclaurin and Jacobi ellipsoids are examples of such motions. In these cases, the ellipsoid rotates as a rigid body about the principal axis (the symmetry axis for the Maclaurin ellipsoid and the shortest axis for the Jacobi ellipsoid).

The Dedekind ellipsoid offers another example of such motions, the axes being invariable in both their lengths and directions. As noted above, the Dedekind ellipsoid is reciprocal to the Jacobi ellipsoid in terms of Theorem 1 (while the Maclaurin ellipsoid is self-reciprocal).

For all the above-mentioned solutions (the Maclaurin, Jacobi, and Dedekind ellipsoids), two pairs of components of the vectors $\boldsymbol{m}$ and $\boldsymbol{\mu}$ vanish, the remaining components being constant (for example, it can be assumed without loss of generality that $m_1 = \mu_1 = m_2 = \mu_2 = 0, m_3 = \mathrm{const}, \mu_3 = \mathrm{const}$).

Riemann [3] has proved the following, more general result:

Theorem 2. *Let* (56) *be satisfied and let all the* A_i *be different. Then* $\boldsymbol{m}$ *and* $\boldsymbol{\mu}$ *are time-independent and at least one pair of components of these vectors vanishes* (*i. e.* $m_i = \mu_i = 0$ *for some* i).

As a consequence, we find that any motion of a shape-preserving fluid ellipsoid whose axes do not coincide, is a fixed point of the reduced system (46) or, which is the same, of the Riemann equations (27). Another proof of this statement is given in [12].

Riemann also noted new solutions — the Riemann ellipsoids — for the case where only one pair of components of $\boldsymbol{m}$ and $\boldsymbol{\mu}$ vanishes (i. e. $m_1 = \mu_1 = 0, \mu_2, m_2, \mu_3, m_3 \neq 0$).

V. A. Stekloff [65, 66] analyzed in detail the case of equality of a pair of axes ($A_i = A_j \neq A_k$) and showed that no shape-preserving motions other than Maclaurin ellipsoids (spheroids) exist in this case. In this sense, he generalized the Riemann result to the axisymmetric case (Riemann himself gave no detailed proof for this case). An attempt of revising Riemann's results was made in [67].

5.2. Axisymmetric Case (Dirichlet [2])

It can easily be shown that the equations of motion determined by the Lagrangian function (38) admit a (two-dimensional) invariant manifold that consists of matrices of the form

$$\mathbf{G} = \begin{Vmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & w \end{Vmatrix},$$

where $\det \mathbf{G} = (u^2 + v^2)w = v_0 = \mathrm{const}$ is the volume of the ellipsoid. This manifold corresponds to an axisymmetric motion of the fluid ellipsoid (see [2]). In this case, the matrix of the principal semiaxes is

$$\mathbf{A} = (\mathbf{G}\mathbf{G}^T)^{1/2} = \mathrm{diag}(\sqrt{u^2+v^2}, \sqrt{u^2+v^2}, w).$$

In view of the condition $\det \mathbf{G} = v_0$, we make the substitution of variables

$$u = v_0^{1/3} r \cos\psi, \quad v = v_0^{1/3} r \sin\psi, \quad w = \frac{v_0^{1/3}}{r^2}$$

and find that the Lagrangian function (38) is

$$L = v_0^{2/3}\left(\left(1 + \frac{2}{r^6}\right)\dot{r}^2 + r^2\dot{\psi}^2 + U_s\right),$$

where

$$U_s = -\frac{2\varepsilon}{v_0}\int\limits_0^\infty \frac{d\lambda}{(\lambda + r^2)\sqrt{\lambda + 1/r^4}} = -\frac{2\varepsilon}{v_0} r^2 \times \begin{cases} \dfrac{2\mathrm{arctg}\sqrt{r^6-1}}{\sqrt{r^6-1}}, & r > 1, \\ \dfrac{\ln\left(\frac{1+\sqrt{1-r^6}}{1-\sqrt{1-r^6}}\right)}{\sqrt{1-r^6}}, & r < 1. \end{cases}$$

The variable ψ is cyclic; therefore, we have a first integral of the form

$$p_\psi = \frac{1}{v_0^{2/3}} \frac{\partial L}{\partial \dot{\psi}} = 2r^2 \dot{\psi},$$

which coincides within a multiplier with the single nonzero component of the momentum M'_{12} (32). With the use of the energy integral (34), we obtain a quadrature that specifies the evolution of r:

$$\left(1 + \frac{2}{r^6}\right) \dot{r}^2 = h - U_*, \quad U_* = U_s + \frac{c}{r^2},$$

where $h = \frac{\mathcal{E}}{m v_0^{2/3}}$ and $c = \frac{p_\psi}{4}$ are fixed values of the energy and momentum integrals. The minimum of the reduced potential U_* corresponds to the Maclaurin spheroid.

5.3. Riemannian Case [3]

There is an invariant manifold more general than the above-described one. It is specified by the block–diagonal matrix of the general form

$$\mathbf{G} = \begin{Vmatrix} u_1 & v_1 & 0 \\ u_2 & v_2 & 0 \\ 0 & 0 & w_3 \end{Vmatrix}. \tag{57}$$

We compute the integrals (30) and (32) obtaining

$$M'_{12} = u_1 \dot{u}_2 - u_2 \dot{u}_1 + v_1 \dot{v}_2 - v_2 \dot{v}_1, \quad M'_{23} = M'_{13} = 0,$$
$$\xi'_{12} = u_1 \dot{v}_1 - v_1 \dot{u}_1 + u_2 \dot{v}_2 - v_2 \dot{u}_2, \quad \xi'_{23} = \xi'_{13} = 0,$$

It is also obvious that $\mathrm{i}\mathbf{Q}$ and $\mathbf{\Theta}$ have in this case a block–diagonal form similar to (57); therefore, this case corresponds to that noted by Riemann, for which, in equations (46), we should set

$$m_1 = m_2 = 0, \qquad m_3 = \text{const},$$
$$\mu_1 = \mu_2 = 0, \qquad \mu_3 = \text{const}.$$

Thus, we obtain a Hamiltonian system with two degrees of freedom, which describes the evolution of the principal semiaxes A_1 and A_2; its Hamiltonian is

$$H = \frac{1}{2} \frac{A_3^{-2}(p_1^2 + p_2^2) + (p_1 A_2^{-1} - p_2 A_1^{-1})^2}{\sum A_i^{-2}} + U_*(A_1, A_2), \tag{58}$$

where the reduced potential is

$$U_* = U_e + \frac{c_1^2}{(A_1 - A_2)^2} + \frac{c_2^2}{(A_1 + A_2)^2},$$

and $c_1^2 = \frac{1}{4}(m_3 + \mu_3)^2$, $c_2^2 = \frac{1}{4}(m_3 - \mu_3)^2$ are fixed constants of the integrals.

The particular version of the system (58) for $c_1 = c_2 = 0$ (i. e., for invariable directions of the principal axes of the ellipsoid) was also noted by Kirchhoff [11], who suggested that the problem does not reduce to quadratures.

At $U_* = 0$, the Hamiltonian (58) describes a geodesic flow on the cubic $A_1 A_2 A_3 = \text{const}$. This remarkable analogy between two different dynamical systems was also noted by Riemann.

5.4. Elliptic Cylinder (Lipschitz [13])

This case can be obtained through a limiting process in the Riemannian case, with one axis of the ellipsoid going to infinity ($A_3 \to \infty$). It is, however, more convenient to start with considering the case of a two-dimensional motion of fluid assuming that the matrix $\mathbf{F}$ has the form

$$\mathbf{F} = \left\| \begin{array}{c|c} \bar{\mathbf{F}} & 0 \\ \hline 0 & 1 \end{array} \right\|, \qquad \det \bar{\mathbf{F}} = 1, \tag{59}$$

where $\bar{\mathbf{F}}$ is a 2×2 matrix with unit determinant.

Obviously, the considerations on which the derivation of the Dirichlet equations [31] was based can be applied to this case without modifications; only the right-hand side of the equations should be properly changed. To this end, it is necessary to use the well-known representation of the potential of the interior points of the elliptic cylinder with a large length l in the system of principal axes

$$U(\boldsymbol{\zeta}) = \bar{\varepsilon}\left(U_0(l) - \frac{\zeta_1^2}{A_1(A_1 + A_2)} - \frac{\zeta_2^2}{A_2(A_1 + A_2)}\right) + O(1/l),$$

where $\bar{\varepsilon} = G\bar{m}$, G is the gravitational constant and $\bar{m} = \pi\rho A_1 A_2$ is the mass per unit length of the cylinder. The constant $U_0(l) \underset{l\to\infty}{\longrightarrow} \infty$ does not appear in the equations of motion and can be omitted.

By analogy with the above considerations, we pass to the Lagrangian representation and make the substitution $\bar{\mathbf{G}} = \bar{\mathbf{F}}\bar{\mathbf{A}}_0$, where $\bar{\mathbf{A}}_0 = \mathrm{diag}(A_1^0, A_2^0)$, to obtain the Lagrangian of the system in the form

$$L = \frac{1}{2}\,\mathrm{Tr}\left(\dot{\bar{\mathbf{G}}}\dot{\bar{\mathbf{G}}}^T\right) - \bar{U}_e,$$
$$\bar{U}_e = -2\bar{\varepsilon}\ln(A_1 + A_2)^2 = -2\bar{\varepsilon}\ln(\mathrm{Tr}(\bar{\mathbf{G}}\bar{\mathbf{G}}^T) + 2\det\bar{\mathbf{G}}).$$

Based on the singular decomposition of the matrix $\bar{\mathbf{G}} = \bar{\mathbf{Q}}^T\bar{\mathbf{A}}\bar{\mathbf{\Theta}}$ with

$$\bar{\mathbf{Q}} = \left\| \begin{matrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{matrix} \right\|, \quad \bar{\mathbf{\Theta}} = \left\| \begin{matrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{matrix} \right\|, \quad \mathbf{A} = \left\| \begin{matrix} A_1 & 0 \\ 0 & A_2 \end{matrix} \right\|,$$

explicitly substituted, we obtain a Lagrangian function in the form

$$L = \frac{1}{2}\left(\dot{A}_1^2 + \dot{A}_2^2 + (A_1\dot{\phi} - A_2\dot{\psi})^2 + (A_2\dot{\phi} - A_1\dot{\psi})^2\right) - \bar{U}_e(A_1, A_2).$$

We can see that the variables ϕ and ψ are cyclic, and there are two linear integrals

$$\frac{\partial L}{\partial\dot{\phi}} = p_\phi, \quad \frac{\partial L}{\partial\dot{\psi}} = p_\psi. \tag{60}$$

We parametrize the relationship $A_1 A_2 = \bar{v}_0$ using hyperbolic functions,

$$A_1 = \bar{v}_0^{1/2}(\mathrm{ch}\,u + \mathrm{sh}\,u), \quad A_2 = \bar{v}_0^{1/2}(\mathrm{ch}\,u - \mathrm{sh}\,u).$$

We use the energy integral and the integrals (60) to obtain a quadrature for the variable u:

$$\bar{v}_0(\mathrm{ch}\,2u)\dot{u}^2 = h - \bar{U}_*,$$
$$\bar{U}_* = 2\bar{\varepsilon}\ln(\mathrm{ch}\,u) + \frac{\bar{c}_1^2}{\mathrm{ch}^2 u} + \frac{\bar{c}_2^2}{\mathrm{sh}^2 u},$$

where $\bar{c}_1^2 = \frac{1}{16}(p_\phi - p_\psi)^2$, $\bar{c}_2^2 = \frac{1}{16}(p_\phi + p_\psi)^2$, and h are fixed constants of the first integrals.

6. CHAOTIC OSCILLATIONS OF A THREE-AXIAL ELLIPSOID

Let us consider in more detail the oscillations (pulsations) of a fluid ellipsoid in the Riemannian case (57). We now represent the equations of motion of the system (58) in a Hamiltonian form most convenient for a numerical investigation of the system. We parametrize the surface $A_1A_2A_3 = v_0$ using cylindrical coordinates

$$\begin{gathered} A_1 = r\cos\phi, \quad A_2 = r\sin\phi, \quad A_3 = \frac{2v_0}{r^2\sin^2 2\phi}, \\ p_1 = p_r\cos\phi - \frac{p_\phi}{r}\sin\phi, \quad p_2 = p_r\sin\phi - \frac{p_\phi}{r}\cos\phi \end{gathered}. \tag{61}$$

The Hamiltonian (58) can be represented in the form

$$H = \frac{1}{2}\left(1 + \frac{c_0^2}{r^6\sin^4 2\phi}\right)^{-1}\left(p_r^2 + \frac{p_\phi^2}{r^2} + \frac{c_0^2}{r^6\sin^4 2\phi}\left(p_r\cos 2\phi - \frac{p_\phi}{r}\sin 2\phi\right)^2\right) + U_*(r,\phi), \tag{62}$$

where $c_0 = 4v_0$.

Since the original system is defined in the quadrant $A_1 > 0, A_2 > 0, A_3 > 0$, for this case we have $0 < \phi < \pi/2$. In this system, the transformation of variables

$$\rho = r^2, \quad \psi = 2\phi, \tag{63}$$

enables obtaining the Hamiltonian in the form

$$H = \frac{2(\rho^2(c_0^2\cos^2\psi + \rho^3\sin^4\psi)p_\rho^2 + \sin^2\psi(c_0^2 + \rho^3\sin^2\psi)p_\phi^2 - 2\rho c_0^2\cos\psi\sin\psi p_\psi p_\phi)}{\rho(c_0^2 + \rho^3\sin^4\psi)} + U_*(\rho,\psi). \tag{64}$$

Upon passing to new Cartesian coordinates according to the formulas

$$x = \rho\cos\psi, \quad y = \rho\sin\psi, \tag{65}$$

we obtain

$$H = 2\rho\left(p_x^2 + \frac{y^4p_y^2}{y^4 + \bar{c}_0^2\rho}\right) + U_*(x,y), \tag{66}$$

where $\rho = \sqrt{x^2+y^2}$; obviously, the system (66) is defined in the upper semiplane ($y > 0$). In this case, as we can see, the kinetic energy of the system has the simplest form.

Remark 7. The transformation (63) is the Levi–Civita transformation (known also as the Bolin transformation) known in celestial mechanics, which is usually written in the complex form

$$x + iy = \rho e^{i\psi} = (A_1 + iA_2)^2.$$

As already noted above, at $U_* = 0$, the Hamiltonian (66) describes a geodesic flow on the cubical surface $A_1A_2A_3 = \text{const}$, embedded in Euclidian space $\mathbb{R}^3$. Almost all trajectories (geodesics) of this system are not compact; therefore, computer simulations for a numerical proof on nonintegrability at $U_* = 0$ cannot be done. As shown recently by S. L. Ziglin [68], this system (at $U_* = 0$, i. e., a geodesic flow) does not admit a meromorphic additional integral.

Remark 8. Various algebraic surfaces in three-dimensional space and singular lines on them (asymptotic lines, lines of curvature, and geodesics) were actively investigated by mathematicians in the 19th century. They were highly enthused by the integrability of the problem of geodesics on the ellipsoid and, generally, on quadrics, discovered by Jacobi. This integrability also refers to multidimensional cases. The problem at hand is a classical example of the separation of variables. Extensive literature has been dedicated to studying this problem from both analytical (integration using theta functions) and qualitative standpoints. However, the mathematicians of the 19th century succeeded little, and nearly nothing was added in the 20th century to finding geodesic flows in higher-order surfaces. Likely, in the context of this problem, Darbu developed a theory of orthogonal families of surfaces investigating, in this avenue, interesting surfaces of the third (Darbu cyclides) and fourth degree (which were also discovered by Roberts [69] and Wangerin [70]). A distinctive property of these families of surfaces is in the fact that they are Lamé families, form a triorthogonal network in three-dimensional space, and a solution for asymptotic and curvature lines can be written for them in the form of elliptic quadratures. However, geodesics for these surfaces have not been found. As our numerical simulations show, the reason for this fact is the nonintegrability of the geodesic flow. In the context of this problem, we also note studies by Schläfli [71] and Cayley [72] dedicated to the classification of various cubic surfaces in space, which, following the fundamental study by Jacobi on geodesics in quadrics, laid the foundation of modern algebraic geometry. Recently, V. V. Kozlov [73] found topological obstacles for the integrability of geodesic flows on noncompact algebraic surfaces (in particular, of the third and fourth degrees). Unfortunately, his results do not apply to Riemannian surfaces $A_1A_2A_3 = \text{const}$.

Shown in Fig. 2 are phase portraits of the system (66). As the plane of the Poincaré map, the plane $x = 1$ is chosen. It can be seen from the diagrams that the phase portrait is virtually regular at energies close to the minimum energy (see Figs 1a and 1c). As the energy is increased, the phase portrait becomes chaotized, which can be clearly seen from Figs 1b and 1d. In Figs 1e and 1f, the intersection of unstable invariant manifolds (separatrices) is also depicted, which can serve as a numerical proof of the nonintegrability of this system.

Remark 9. In the numerical integration of the equations of motion, it is convenient to use the representation of the potential U_e (34) and its derivatives in terms of elliptic functions,

$$I(A_1, A_2, A_3) = \int\limits_0^\infty \frac{d\lambda}{\Delta(\lambda)} = \frac{2}{æ} F(\varphi, k),$$

$$æ = \sqrt{A_1^2 - A_3^2}, \quad k = \sqrt{\frac{A_1^2 - A_2^2}{A_1^2 - A_3^2}}, \quad \varphi = \arcsin\left(\sqrt{\frac{A_1^2 - A_3^2}{A_1^2}}\right),$$

where $F(\varphi, k) = \displaystyle\int_0^\varphi \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$ is the first-kind elliptic integral.

Due to the invariance with respect to permutations of the quantities A_1, A_2, and A_3, they can be ordered so as to make all values of $æ, \varphi$, and k real. To find the derivatives (which are obviously not invariant with respect to permutations of A_i), a necessary ordering of the quantities A_1, A_2, and A_3 should be made first, after which the derivative of the integral I with respect to the corresponding argument should be calculated.

For the derivatives $F(\varphi, k)$, the following relationships are valid:

$$\frac{\partial F}{\partial \varphi} = \left(1 - k^2 \sin^2 \varphi\right)^{-1/2},$$

$$\frac{\partial F}{\partial k} = \frac{1}{1 - k^2}\left(\frac{E(\varphi, k) - (1 - k^2)F(\varphi, k)}{k} - \frac{k \sin\varphi \cos\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}\right),$$

where $E(\varphi, k) = \displaystyle\int_0^\varphi \sqrt{1 - k^2 \sin^2 \alpha}\, d\alpha$ is the second-kind elliptic integral.

II. DYNAMICS OF A GAS CLOUD WITH ELLIPSOIDAL STRATIFICATION

1. INTRODUCTION

The investigation of the dynamics of gas ellipsoids traces back to a study by L. V. Ovsyannikov [74] (1956), who analyzed the most general equations describing the motion of an ideal polytropic gas, without taking into account gravitation, with a velocity field linear in the coordinates of the gas particles (from here on, by the gas ellipsoid, we mean the Dirichlet solution generalized to various models of compressible fluid). Note that the paper [74] is very brief and purely mathematical:

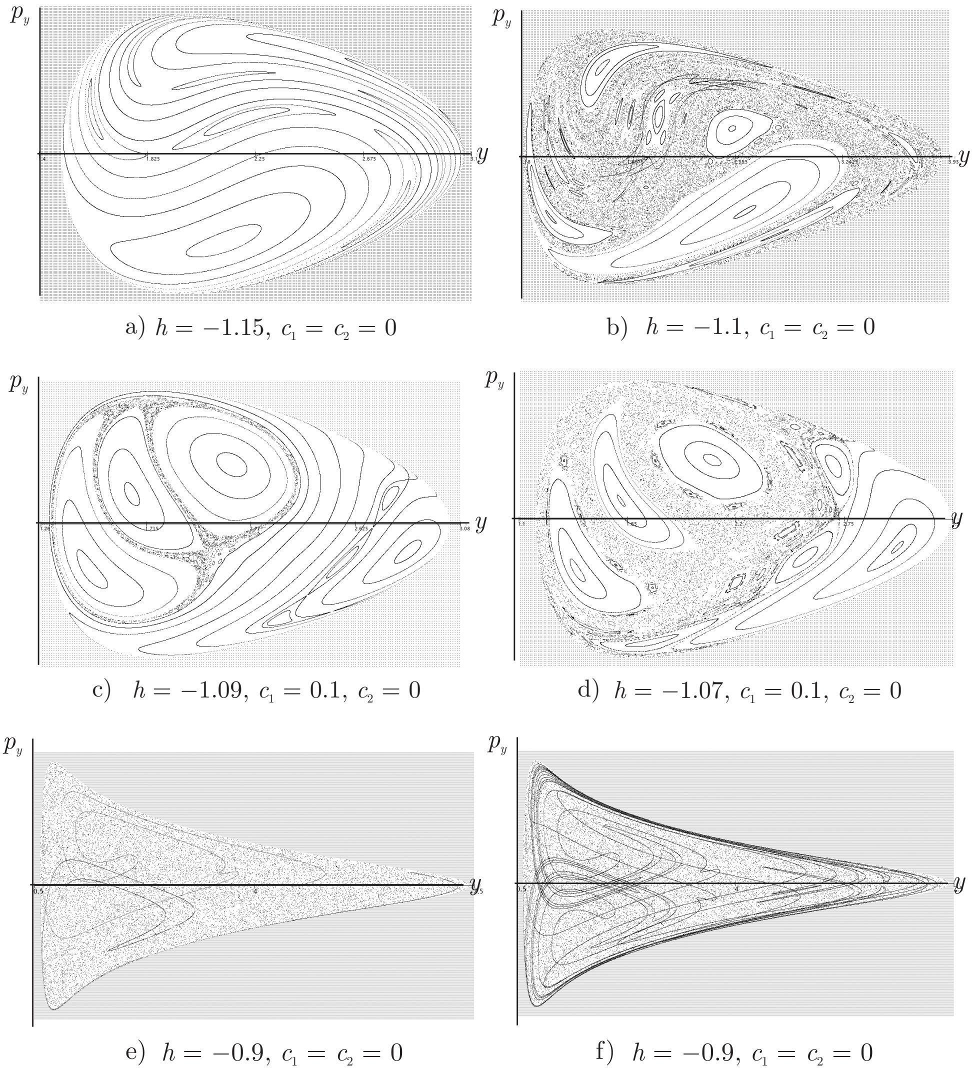

Fig. 2. The Poincaré map of system (66). For all panels, $c_0 = 1, \varepsilon = 0,6$; for the map, the planes $x = 1$ (a–d) and $x = 0.1$ (e ,f) are chosen.

in fact, the equations of motion are obtained there, several possible cases of the existence of the considered solution are noted, and an incomplete set of first integrals is given for two cases. It is interesting that Ovsyannikov's paper contains no references, so that the relationship between the obtained solution and the Dirichlet solution is not revealed.

Later, D. Lynden-Bell [75] (1962) demonstrated, also without any references, the existence of the solution in the form of a spheroid for a self-gravitating dust cloud (i. e., for a medium not resisting to deformations, $p \equiv 0$).

Ya. B. Zel'dovich [76] (1965) obtained the equations of motion of a self-gravitating dust ellipsoid in the general case and studied (on a physical level of rigor) the possibility of collapse and expansion in this problem. Likely, Ya. B. Zel'dovich also overlooked the relationship of this problem to the Dirichlet–Riemann problem, since the model of a dust cloud can be obtained simply by setting $p = 0$ in the Dirichlet equations.

Independently of Ovsyannikov (at least without a reference), F. Dyson [77] (1968) obtained the equations of motion of an ideal-gas cloud in the case of an isothermal flow (although without the assumption of a polytropic behavior of the gas); a Gaussian density distribution with an ellipsoidal stratification was found. Dyson noted a relationship between the obtained solution and the Dirichlet problem and wrote the equations of motion of the gas ellipsoid in a Riemannian form.

Also independently of Ovsyannikov, M. Fujimoto [78] (1968) describes a model of a cooling ellipsoidal gas cloud; in essence, he obtains a generalization of a case considered by Ovsyannikov (if we assume the cooling parameter to be æ = 0, we will obtain Ovsyannikov's equations). In addition, in Fujimoto's model, the density is constant, which enabled taking into account the gravitational interaction between the particles of the cloud. Fujimoto also noted a relationship of this problem to the Dirichlet problem and, in studying it, used the techniques developed by Chandrasekhar [1] and Rossner [79].

Let us also mention a study by Anisimov [80] (1970), who follows [74] and [77] considering two cases of the integrable dynamics of a gas ellipsoid without allowances for gravitation but with the additional condition of the monoatomic structure of the gas (a polytropic index of $\gamma = \frac{5}{3}$). The first case is the motion of an axisymmetric ellipsoid; the second, of an elliptic cylinder. A nonautonomic Jacobi integral was found (which is due to the uniformity of the potential with a uniformity degree of -2). This integral is essentially necessary for integration in the cases under study; as we will show below, these systems are not integrable in the general case.

Bogoyavlenskiĭ [81] (1976) analyzes the dynamics of a gas ellipsoid on a physical level of rigor taking into account gravitation (i. e., he considers the Fujimoto model without cooling). Explicit Lagrangian and Hamiltonian representations of the system are used.

Gaffet [83–85] shows that the system that describes irrotational gas ellipsoids without considering gravitation, for a monoatomic gas $\left(\gamma = \frac{5}{3}\right)$, satisfies the Painlevé property; in these studies, first integrals are found and integration in quadratures is carried out for certain particular cases.

There are also studies analyzing a spherically symmetric motion of a gas cloud; one of the most general solutions is described by Lidov [86], who considers time-dependent, one-dimensional, spherically symmetric, adiabatic motions of a self-gravitating mass of a perfect gas.

Nemchinov [87] uses a solution that describes the ellipsoidal expansion of a gas cloud to find characteristic features of nonspherical explosions (in particular, he notes an increase in the impact of the stream in the direction of one of the principal axes compared to a similar spherical explosion); the effect of the heating of the cloud on the expansion speed is also investigated.

Finally, let us mention a series of studies (see [88] and references therein) generalizing the problem of the expansion of an ellipsoidal cloud to vacuum (or the collapse of an ellipsoidal cavity) with the presence of a rarefaction (compression) wave.

2. EQUATIONS OF MOTION OF A GAS CLOUD WITH A LINEAR VELOCITY FIELD

Now consider in a similar way the case where the motion of a *compressible fluid* (*gas*) is also defined by a linear transformation of the Lagrangian coordinates

$$\boldsymbol{x}(\boldsymbol{a}, t) = \mathbf{F}(t)\boldsymbol{a}; \tag{67}$$

for the compressible medium, the condition $\det \mathbf{F} = 1$ is obviously not valid. Clearly, the velocity field is linear in the coordinates of the fluid particles:

$$\boldsymbol{v}(\boldsymbol{x}, t) = \dot{\boldsymbol{x}} = \dot{\mathbf{F}}(t)\mathbf{F}^{-1}(t)\boldsymbol{x}.$$

In this case, the equations that describe the flow in the given case (for potential forces) have the following Lagrangian representation:

$$\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^T \ddot{\boldsymbol{x}} = -\frac{\partial U}{\partial \boldsymbol{a}} - \frac{1}{\rho}\frac{\partial p}{\partial \boldsymbol{a}}, \tag{68}$$

and the continuity equation in the Lagrangian form is

$$\dot{\rho} + \rho \operatorname{Tr}\left(\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^{-1} \frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{a}}\right) = 0; \tag{69}$$

here, $\operatorname{Tr}\left(\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^{-1} \frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{a}}\right) = \operatorname{div} \boldsymbol{v}(\boldsymbol{x}, t)$.

For the flow of the structure under study (67), the continuity equations can easily be integrated. Indeed, if we introduce the notation $\varphi(t) = \det \mathbf{F}(t)$ using the relationship $\left(\frac{\partial\varphi}{\partial\mathbf{F}}\right)^T = \varphi\mathbf{F}^{-1}$, we find that $\operatorname{Tr}\left(\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^{-1}\frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{a}}\right) = \frac{\dot\varphi}{\varphi}$; therefore,

$$\rho(\boldsymbol{a}, t) = \frac{f(\boldsymbol{a})}{\varphi(t)}, \tag{70}$$

where the function $f(\boldsymbol{a})$ is time-independent.

Except the four functions $\boldsymbol{x}(\boldsymbol{a}, t), p(\boldsymbol{a}, t)$, the medium at hand is described by three additional scalar quantities — the density $\rho(\boldsymbol{a}, t)$, the specific internal energy $U_{\text{in}}(\boldsymbol{a}, t)$, and the temperature $T(a, t)$. Therefore, it is necessary to complement the system (68), (69) with three other equations. As is known [89], these additional equations are of a thermodynamic rather than mechanical nature and depend essentially on our assumptions concerning the properties of the medium and on the character of the flow.

Depending on these assumptions, various gasdynamic models can be obtained. Consider three of them that are most widely known, emphasizing the explicit assumptions. Unless the opposite is stipulated, we assume in what follows that the potential of the external forces applied to the system, U, is zero.

2.1. The Ovsyannikov Model [74]

1°. *The gas is ideal and can be described by the equation of state*

$$p = \rho RT, \tag{71}$$

where R is the universal gas constant.

2°. *The gas is polytropic, and its internal energy depends linearly on the temperature,*

$$U_{\text{in}} = c_V T, \tag{72}$$

where $c_V = \text{const}$ is the specific heat at constant volume.

3°. *The gas flow is adiabatic (i. e., there is no heat exchange between different parts of the gas volume); therefore, the energy variations are described by the equation*

$$\dot U_{\text{in}} = -p\left(\frac{1}{\rho}\right)^{\boldsymbol{\cdot}}. \tag{73}$$

Remark 10. Equation (73) is a consequence of the first principle of thermodynamics, $\delta Q = dU_{\text{in}} + p\,dV$, where, in view of assumption 3°, it is necessary to set $\delta Q = 0$, $(V = \frac{1}{\rho})$.

Remark 11. Recall that, due to the well-known thermodynamic identity

$$\left(\frac{dU_{\text{in}}}{dV}\right)_T = \left(T\left(\frac{\partial p}{\partial T}\right)_V - p\right)$$

the internal energy of the ideal gas (71) depends only on the temperature, $U_{\text{in}} = U_{\text{in}}(T)$.

We find using equations (71)–(73) and taking into account the relationship (70) that

$$\frac{\dot p}{p} + \gamma\frac{\dot\varphi}{\varphi} = 0,$$

where the dimensionless constant $\gamma = 1 + \frac{R}{c_V}$ is the adiabatic index. Thus, for the thermodynamic quantities, we have

$$p(\boldsymbol{a}, t) = \frac{g(\boldsymbol{a})}{\varphi^\gamma(t)}, \quad U_{\text{in}}(\boldsymbol{a}, t) = \frac{1}{\gamma - 1}RT(\boldsymbol{a}, t) = \frac{1}{\gamma-1}\varphi^{1-\gamma}(t)\frac{g(\boldsymbol{a})}{f(\boldsymbol{a})},$$

where $g(\boldsymbol{a})$ is an arbitrary, time-independent quantity.

Thus, for the existence of a solution of the form (67) in this case, it should necessarily be required that

$$\frac{1}{f(\boldsymbol{a})}\boldsymbol{\nabla}_a g(\boldsymbol{a}) = \mathbf{V}\boldsymbol{a}, \tag{74}$$

where $\mathbf{V}$ is a certain constant matrix, $\boldsymbol{\nabla}_a = \left(\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3}\right)$. Then the equations of motion for $\mathbf{F}(t)$ are

$$\mathbf{F}^T\ddot{\mathbf{F}} + (\det\mathbf{F})^{1-\gamma}\mathbf{V} = 0 \qquad (\textit{the Ovsyannikov equations}). \tag{75}$$

As Ovsyannikov has shown [74], it is sufficient to consider the following solutions of equation (74).

Theorem 3. *Any solution of equation* (74) *reduces via a linear transformation of the Lagrangian coordinates to one of the following four types* (*depending on the rank of the matrix* $\mathbf{V}$)*:*

(I) $\mathbf{V} = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\varepsilon_i = \pm 1 \quad (i = 1, 2, 3)$,
$g(\boldsymbol{a}) = g(s), \quad f(\boldsymbol{a}) = 2g'(s), \quad s = (\boldsymbol{a}, \mathbf{V}\boldsymbol{a})$;

(II) $\mathbf{V} = \begin{Vmatrix} \varepsilon_1 & \delta & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad \varepsilon_i = \pm 1 \quad (i = 1, 2)$,
if $\delta \neq 0$ *then* $g(\boldsymbol{a}) = g(s), \quad f(\boldsymbol{a}) = \frac{a_1 g'(s)}{(\boldsymbol{a}, \mathbf{V}\boldsymbol{a})}, \quad s = a_1 \bar{s}\left(\frac{a_2}{a_1}\right), \quad \ln \bar{s}(\lambda) = \int \frac{\varepsilon_2 \lambda\, d\lambda}{\varepsilon_1 + \delta\lambda + \varepsilon_2\lambda}$;
if $\delta = 0$ *then* $g(\boldsymbol{a}) = g(s), \quad f(\boldsymbol{a}) = 2g'(s), \quad s = (\boldsymbol{a}, \mathbf{V}\boldsymbol{a})$;

(III) $\mathbf{V} = \mathrm{diag}(\varepsilon, 0, 0)$, $\varepsilon = \pm 1, \quad g(\boldsymbol{a}) = g(a_1), \quad f(\boldsymbol{a}) = \frac{\varepsilon}{a_1} g'(a_1)$;

(IV) $\mathbf{V} = 0, \quad g(\boldsymbol{a}) = \mathrm{const}, \quad f(\boldsymbol{a})$ *is an arbitrary function.*

Proof. We briefly note the principal steps in the proof. Rewriting the solvability condition (74) in the form $\boldsymbol{\nabla}_a \times \boldsymbol{\nabla}_a g(\boldsymbol{a}) = \boldsymbol{\nabla}_a \times f(\boldsymbol{a})\mathbf{V}\boldsymbol{a} = 0$ yields

$$\boldsymbol{\nabla}_a f(\boldsymbol{a}) \times (\mathbf{V}\boldsymbol{a}) = f(\boldsymbol{a})\boldsymbol{\omega}_V, \tag{76}$$

where $\boldsymbol{\omega}_V = (V_{23} - V_{32}, V_{31} - V_{13}, V_{12} - V_{21})$. We take the dot product by $\mathbf{V}\boldsymbol{a}$ and represent the solvability condition as the vector equation

$$\mathbf{V}^T\boldsymbol{\omega}_V = 0. \tag{77}$$

In addition, it is clear that equations (75) are invariant with respect to nonsingular transformations of the Lagrangian variables, for which

$$\boldsymbol{a} = \mathbf{S}\boldsymbol{a}', \quad \mathbf{V}' = \mathbf{S}^T\mathbf{V}\mathbf{S}, \quad \mathbf{F}' = (\det\mathbf{S})^{\frac{1-\gamma}{1+\gamma}}\mathbf{F}\mathbf{S}, \quad \det\mathbf{S} \neq 0. \tag{78}$$

If $\mathrm{rank}\,\mathbf{V} < 3$, the most general matrix of the corresponding rank reduces via the transformations (78) to the form given in cases II–IV. If $\mathrm{rank}\,\mathbf{V} = 3$, then $\boldsymbol{\omega}_V = 0$; therefore, $\mathbf{V}$ is symmetric and reduces to the form of case I.

The relationships (74) and (76) make it possible to easily find the corresponding functions $f(\boldsymbol{a})$ and $g(\boldsymbol{a})$ from the known matrix $\mathbf{V}$, □

From the physical standpoint, case I with a sign-definite matrix $\mathbf{V}$ is most interesting. In particular, if we set $\mathbf{V} = \mathrm{diag}(-1, -1, -1)$ (i. e., $s = -(\boldsymbol{a}, \boldsymbol{a})$) and choose a linear function $g(s)$, we will obtain

$$g(\boldsymbol{a}) = \frac{1}{2}\rho_0(d_0^2 - (\boldsymbol{a}, \boldsymbol{a})), \quad f(\boldsymbol{a}) = \rho_0, \tag{79}$$

where we must assume that $\rho_0 > 0$ (since the density of the gas is positive). In this case, the gas is distributed with a constant density $\rho = \rho_0/\det(\mathbf{F})$ inside the finite ellipsoidal volume

$$(\boldsymbol{a}, \boldsymbol{a}) = (\boldsymbol{x}, (\mathbf{F}\mathbf{F}^T)^{-1}\boldsymbol{x}) \leqslant d_0^2. \tag{80}$$

Therefore, the solution of the form (67) in this case remains valid upon adding gravitational forces (the matrix (21) with $\mathbf{A}_0 = \mathbf{E}$ being added to the right-hand side of equation (75)). The problem of the motion of a compressible self-gravitating gas cloud was formulated in [81]. However, the analysis presented there appears to be somewhat naive in view of the modern knowledge of regular and chaotic motions in dynamical systems.

2.2. The Dyson Model [77]

Assumptions 1° *and* 3° *coincide with those in the preceding case, whereas, instead of the polytropic behavior, we assume that*

2°*. The gas is isothermal at the initial time, i. e.,* $T(\boldsymbol{a}, t = 0)$ *does not depend on* $\boldsymbol{a}$*.*

We substitute the pressure from the equation of state (71) into (73) and make use of (70) to obtain

$$\frac{\dot{U}_{\text{in}}}{RT} = -\frac{\dot{\varphi}}{\varphi}. \tag{81}$$

At the same time, as mentioned above (see Note 9), the internal energy depends only on the temperature, and the right-hand side of (81) does not depend on $\boldsymbol{a}$; therefore, the gas remains isothermal at all later times and (81) can be represented as

$$\varphi\frac{dU_{\text{in}}}{d\varphi} + RT = 0. \tag{82}$$

By integrating this equation in view of (72), we obtain a relationship between T and φ:

$$\varphi = \varphi_0 \exp\left(-\int (RT)^{-1}\left(\frac{dU_{\text{in}}}{dT}\right)dT\right). \tag{83}$$

Thus, according to (70), (71), and (82), the pressure can ultimately be written as

$$p(\boldsymbol{a}, t) = \frac{RT(\varphi(t))}{\varphi(t)} f(\boldsymbol{a}). \tag{84}$$

We substitute (84) into (68) and restrict ourselves to the case where no external forces are present (i. e., $U = 0$). Thus, we find that the existence of a solution of the form (67) requires that $\ln f(\boldsymbol{a})$ be a uniform quadratic function of the Lagrangian coordinates. Since the Lagrangian coordinates are defined to within a nonsingular linear substitution (78), we can represent $f(\boldsymbol{a})$ in the form

$$f(\boldsymbol{a}) = \frac{m}{(2\pi)^{3/2}}\exp(-\frac{1}{2}(\boldsymbol{a}, \boldsymbol{a})), \tag{85}$$

where $m = \int \rho(\boldsymbol{x})d^3\boldsymbol{x} = \int f(\boldsymbol{a})d^3\boldsymbol{a}$ is the mass of the gas.

Finally, for the elements of the matrix $\mathbf{F}$, we obtain the equation of motion

$$\mathbf{F}^T\ddot{\mathbf{F}} = RT(\varphi)\mathbf{E} \qquad (\textit{the Dyson equations}). \tag{86}$$

Remark 12. According to (85), we find that the gas, during its motion, has an ellipsoidal density stratification of the form

$$\rho(\boldsymbol{x}, t) = \frac{1}{\varphi(t)} f(n^2), \quad n^2 = (\boldsymbol{x}, (\mathbf{F}\mathbf{F}^T)^{-1}\boldsymbol{x}). \tag{87}$$

Thus, everywhere on the ellipsoid $n^2 = \text{const}$, the density ρ has the same value. The gravitational potential of such bodies, as is known, is not a uniform quadratic function of the coordinates; therefore, no solution of the form (67) exists with the presence of gravitation.

2.3. Model of a Cooling Gas Cloud (Fujimoto [78])

In this model, the assumptions 1° and 2° coincide with those in Ovsyannikov's case, i. e., the gas is assumed to be ideal and polytropic, while the third assumption in this case has the form

3°. *The motion of the gas is not adiabatic, the variations in the internal energy satisfying the equation*

$$\rho\dot{U}_{\rm in} + p\,\mathrm{Tr}\left(\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^{-1}\frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{a}}\right) = -æ\rho^n T^m. \tag{88}$$

Remark 13. Equations (88) differ from the equations of an adiabatic process (73) by the terms $-æ\rho^n T^m$.

We use equations (70)–(72) to eliminate $U_{\rm in}$ from equation (88) and find

$$\frac{\dot{p}}{p} + \gamma\frac{\dot{\varphi}}{\varphi} = -\frac{æ(\gamma-1)}{R}\rho^{n-1}T^{m-1}. \tag{89}$$

To obtain a solution in the form (67), we additionally require that

$$m = 1, \quad \rho(\boldsymbol{a}, t) = \frac{\rho_0}{\varphi(t)},$$

where $\rho_0 = \mathrm{const}$ is independent of $\boldsymbol{a}$, i. e., the density is constant inside the cloud. The solution of equation (89) in this case has the form $p(\boldsymbol{a}, t) = \sigma(t)g(\boldsymbol{a})$, where $\sigma(t)$ satisfies the equation

$$\frac{\dot{\sigma}}{\sigma} + \gamma\frac{\dot{\varphi}}{\varphi} = -\bar{æ}\varphi^{1-n}, \quad \bar{æ} = \frac{æ(\gamma-1)}{R}\rho_0^{n-1}. \tag{90}$$

The function $g(\boldsymbol{a})$ should obviously satisfy equation (74), and it can easily be shown that, according to Theorem 3, we may choose

$$g(\boldsymbol{a}) = 1 - (\boldsymbol{a}, \boldsymbol{a}), \quad f(\boldsymbol{a}) = \mathrm{const}.$$

The condition of the finiteness of the total gas mass implies that $\mathbf{V} = \mathrm{diag}(-1, -1, -1)$, the gas occupying initially the region $(\boldsymbol{a}, \boldsymbol{a}) \leqslant 1$ (in the original physical variables, this inequality specifies an ellipsoid of the form $(\boldsymbol{x}, (\mathbf{F}, \mathbf{F})^{-1}\,|_{t=0}\,\boldsymbol{x}) \leqslant 1$).

Finally, we obtain the system of equations describing the dynamics of the cooling cloud in the form

$$\mathbf{F}^T\ddot{\mathbf{F}} = \frac{2\sigma\varphi}{\rho_0}\mathbf{E}\left(+2\varepsilon\int\limits_0^\infty \mathbf{F}^T(\mathbf{F}\mathbf{F}^T + \lambda\mathbf{E})^{-1}\mathbf{F}\frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{F}^T + \lambda\mathbf{E})}}\right),$$
$$(\ln(\sigma\varphi^\gamma))^{\cdot} = -\bar{æ}\varphi^{1-n}.$$

The parenthesized term describes the gravitational interaction between the particles of the cloud. Allowances for the gravitational interaction in the solution of the form (68) are possible in this case due to the uniformity of gas in the cloud ($\rho_0 = \mathrm{const}$).

The numerical results of [78] demonstrate a possibility of gravitational collapse in this system (at $æ > 0$).

2.4. Model of a Dust Cloud (Gravitational Collapse)

1°. *The medium* (*dust*) *does not counteract deformations,*

$$p \equiv 0.$$

2°. *At the initial time, the particles are distributed uniformly* (*inside the ellipsoid*),

$$\rho(t, \boldsymbol{a})\big|_{t=0} = \rho_0 = \text{const}.$$

For a solution of the form (68), the density obviously does not depend on the coordinates at all subsequent times, being determined by the relationship

$$\rho(t) = \frac{\rho_0}{\det \mathbf{F}(t)}.$$

Therefore, allowances for the gravitational attraction of particles in the clouds are possible in this model in the framework of the linear solution (68), and the equations of motion can be written as

$$\mathbf{F}^T\ddot{\mathbf{F}} = 2\varepsilon \int\limits_0^\infty \mathbf{F}^T(\mathbf{F}\mathbf{F}^T + \lambda\mathbf{E})^{-1}\mathbf{F}\frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{F}^T + \lambda\mathbf{E})}}. \tag{91}$$

This model is used in astrophysics to describe the gravitational collapse [76]. In particular, it is applied in [75] to the description of the collapse of an elliptic has cloud at zero temperature.

3. LAGRANGIAN FORMALISM, SYMMETRIES, AND FIRST INTEGRALS

We will now show that the Dyson equations (86), the Ovsyannikov equations (75) under the condition (79), and the equations of a dust cloud (91) admit a natural Lagrangian description. It can be shown by means of direct calculations that the equations of motion can be written in the form

$$\begin{gathered} \left(\frac{\partial L}{\partial \dot{\mathbf{F}}}\right)^{\boldsymbol{\cdot}} - \frac{\partial L}{\partial \mathbf{F}} = 0, \\ L = \frac{1}{2}\operatorname{Tr}(\dot{\mathbf{F}}\dot{\mathbf{F}}^T) - U_g(\mathbf{F}), \end{gathered} \tag{92}$$

where

$$U_g(\mathbf{F}) = \begin{cases} U_{\text{in}}(\varphi)\,\textit{for the Dyson model}, \\ \dfrac{1}{\gamma-1}\varphi^{1-\gamma} - 2\varepsilon\displaystyle\int\limits_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{F}^T+\lambda\mathbf{E})}}\,\textit{for the Ovsyannikov model with gravitation}, \\ -2\varepsilon\displaystyle\int\limits_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{F}^T+\lambda\mathbf{E})}}\,\textit{for the dust-cloud model}, \end{cases} \tag{93}$$

where, as above, $\varphi = \det\mathbf{F}, \mathbf{F} \in GL(3)$.

Remark 14. In the Dyson model, the Lagrangian representation (92) can be directly obtained from the Hamiltonian principle for barotropic flows (see [11])

$$\delta\int\limits_{t_1}^{t_2}(T-U)\,dt = \delta\int\limits_{t_1}^{t_2}W\,dt, \tag{94}$$

where T and U are the kinetic and the potential energy of the fluid and W is the barotropic potential that satisfies the equation

$$\delta W = \int p \frac{\delta \rho}{\rho} d^3\boldsymbol{x}. \tag{95}$$

Based on the above assumption, we obtain for our case within a constant:

$$W = \int RT \ln \rho d^3\boldsymbol{x} = U_{\text{in}}. \tag{96}$$

These considerations can also be generalized to the Ovsyannikov model.

By analogy with the fluid ellipsoid (see Section 2, § 4), we conclude that the system (92) is invariant with respect to linear transformations of the form

$$\mathbf{F}' = \mathbf{S}_1 \mathbf{F} \mathbf{S}_2, \quad \mathbf{S}_1, \mathbf{S}_2 \in SO(3), \tag{97}$$

which form a symmetry group $\Gamma = SO(3) \otimes SO(3)$.

The Dedekind reciprocity law (*Teorem* 1 *in Part* 1)*, which corresponds to a discrete transformation* $\mathbf{F}' = \mathbf{F}^T$*, is also valid in the dynamics of gas clouds.*

According to the Noether second theorem, integrals of motion linear in velocity — the vorticity and total angular momentum of the system – correspond to the transformations (97) and can be represented in the matrix form

$$\boldsymbol{\Xi} = \mathbf{F}^T \dot{\mathbf{F}} - \dot{\mathbf{F}}^T \mathbf{F}, \quad \mathbf{M} = \mathbf{F} \dot{\mathbf{F}}^T - \dot{\mathbf{F}} \mathbf{F}^T. \tag{98}$$

In addition, there is also a quadratic integral, the total energy of the system

$$\mathcal{E} = \frac{1}{2} \operatorname{Tr}(\dot{\mathbf{F}} \dot{\mathbf{F}}^T) + U_g(\mathbf{F}). \tag{99}$$

4. SYMMETRY-BASED REDUCTION AND HAMILTONIAN FORMALISM

It is not difficult to carry out a reduction based on the linear integrals (98) using the results of the preceding section. To this end, we make use of the Riemannian decomposition

$$\mathbf{F} = \mathbf{Q}^T \mathbf{A} \boldsymbol{\Theta}, \quad \mathbf{Q}, \boldsymbol{\Theta} \in SO(3), \quad \mathbf{A} = \operatorname{diag}(A_1, A_2, A_3).$$

For the Lagrangian function of the gas cloud (92), in view of the equations

$$\dot{\mathbf{Q}} = \mathbf{w} \mathbf{Q}, \quad \dot{\boldsymbol{\Theta}} = \boldsymbol{\omega} \boldsymbol{\Theta},$$

we obtain the expression

$$L = \frac{1}{2} \sum \dot{A}_i^2 + \frac{1}{4} \sum (A_j + A_k)^2 (\mathrm{w}_i - \omega_i)^2 + (A_j - A_k)^2 (\mathrm{w}_i + \omega_i)^2 - U_g(\mathbf{A}).$$

We denote the three-dimensional vector of semiaxes as $\boldsymbol{q} = (A_1, A_2, A_3)$ and represent the equations of motion in the form

$$\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)^{\cdot} - \frac{\partial L}{\partial \boldsymbol{q}} = 0,$$

$$\left(\frac{\partial L}{\partial \mathrm{w}}\right)^{\cdot} = \frac{\partial L}{\partial \mathrm{w}} \times \mathrm{w}, \quad \left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right)^{\cdot} = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega}.$$

This is an analog of the Riemann equations (16), (40) for the case of a gas cloud (the difference is in the absence of the term containing pressure). These equations can easily be written in a matrix form similar to (16) [77].

The Lagrangian transformation

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}, \quad \boldsymbol{m} = \frac{\partial L}{\partial \dot{\mathrm{w}}}, \quad \boldsymbol{\mu} = \frac{\partial L}{\partial \dot{\boldsymbol{\omega}}}$$

yields a Hamiltonian system

$$\boldsymbol{q} = \frac{\partial H}{\partial \dot{\boldsymbol{p}}}, \quad \boldsymbol{p} = -\frac{\partial H}{\partial \dot{\boldsymbol{q}}}, \quad \dot{\boldsymbol{m}} = \boldsymbol{m} \times \frac{\partial H}{\partial \boldsymbol{m}}, \quad \dot{\boldsymbol{\mu}} = \boldsymbol{\mu} \times \frac{\partial H}{\partial \boldsymbol{\mu}},$$
$$H = \frac{1}{2}\sum p_i^2 + \frac{1}{4}\sum \left(\frac{m_i+\mu_i}{q_j-q_k}\right)^2 + \left(\frac{m_i-\mu_i}{q_j+q_k}\right)^2 + U_g(\boldsymbol{q}). \tag{100}$$

The Poissonian structure of the system (100) has the form

$$\{q_i, p_j\} = \delta_{ij}, \quad \{m_i, m_j\} = \varepsilon_{ijk} m_k, \quad \{\mu_i, \mu_j\} = \varepsilon_{ijk}\mu_k, \tag{101}$$

where the zero brackets are omitted. As above, the bracket (101) has two Casimir functions

$$\Phi_m = (\boldsymbol{m}, \boldsymbol{m}), \quad \Phi_\mu = (\boldsymbol{\mu}, \boldsymbol{\mu}),$$

which correspond to the squared total momentum and the vorticity of the system.

In the general case ($\Phi_m \neq 0$, $\Phi_\mu \neq 0$)*, we have a Hamiltonian system with five degrees of freedom.*

In the particular case of $\Phi_m = 0$ *or* $\Phi_\mu = 0$*, we have a system with four degrees of freedom.*

If $\Phi_m = \Phi_\mu = 0$*, we obtain a system with three degrees of freedom* similar to the problem of the motion of a unit-mass point in $\mathbb{R}^3 = \{\boldsymbol{q}\}$.

5. PARTICULAR CASES OF MOTION

5.1. Case of $\gamma = \frac{5}{3}$ *(Monoatomic Gas)*

Consider, in greater detail, the case of the expansion of an ellipsoidal cloud of ideal monoatomic gas in the absence of gravitation; we will show that the system has additional symmetries in this case, where, as is known, $c_V = \frac{3}{2}R$ and, therefore, $\gamma = \frac{5}{3}$.

We use (92) to represent the Lagrangian of the system as

$$L = \frac{1}{2}\operatorname{Tr}(\dot{\mathbf{F}}\dot{\mathbf{F}}^T) - U_g(\mathbf{F}), \quad U_g(\mathbf{F}) = \frac{3}{2}k\frac{1}{(\det \mathbf{F})^{2/3}}, \tag{102}$$

where $k = \text{const}$ is a positive constant (introduced for convenience). The integrals — vorticity $\boldsymbol{\Xi}$, momentum $\mathbf{M}$, and energy $\mathcal{E}$ — were mentioned above (98), (99).

We denote the eigenvalues of the matrices $\mathbf{F}\mathbf{F}^T$ as A_1^2, A_2^2, A_3^2 and call A_i the principal semiaxes of th gas ellipsoid (A_i coincides with the semiaxes of the gas ellipsoid in Ovsyannikov's model at the pressure and density distribution (79); for Dyson's model with a normal density distribution (85), this term is only conventional). We define an analog of the central moment of inertia of the system by the formula

$$I = \operatorname{Tr}\mathbf{F}\mathbf{F}^T = A_1^2 + A_2^2 + A_3^2. \tag{103}$$

As we can see, according to (102), the dynamics of the could can be described in this case by a natural Lagrangian system with a uniform potential of uniformity degree $\alpha = -2$ (for an arbitrary γ, the uniformity degree is $\alpha = 3(1-\gamma)$). We use the Lagrange–Jacobi formula for uniform systems [82] to obtain

$$\ddot{I} = 4\mathcal{E} = \text{const},$$

where $\mathcal{E}$ is the energy of the system (for an arbitrary γ, we find $\ddot{I} = 4\mathcal{E} - 2(3(1-\gamma)+2)U_g$).

The integration of this relationship yields

$$I = 2\mathcal{E}t^2 + at + b, \tag{104}$$

where the integration constants a and b can be expressed in terms of the phase variables and time according to the formulas

$$a = 2\operatorname{Tr}(\mathbf{F}^T\dot{\mathbf{F}}) - 4\mathcal{E}t, \quad b = 2\mathcal{E}t^2 - 2\operatorname{Tr}(\mathbf{F}^T\dot{\mathbf{F}})t + I. \tag{105}$$

In fact, a and b are nonautonomic (explicitly time-dependent) integrals of the system considered.

For the first time, the integral (104) for uniform systems of a degree of -2 was noted by Jacobi in the problem of the motion of particles in a straight line. For the problem of the motion of a gas cloud, the Jacobi integral was found in [80]. The integrals (105) for system (102) were indicated in [83], while corresponding symmetries in the particular case of $\mathbf{\Xi} = 0$ were mentioned in [84].

Proposition 1. *At $t \to \pm\infty$, at least one of the semiaxes, A_i, of the gas cloud goes to infinity.*

Except the nonautonomic integrals (105), the systems in this case admits an autonomic quadratic integral independent of the energy integral,

$$J = 2I\mathcal{E} - [\mathrm{Tr}(\mathbf{F}^T\dot{\mathbf{F}})]^2. \tag{106}$$

For uniform systems of degree -2, this integral was found in a more general case in [90]. For the system (102) in the particular case of $\mathbf{\Xi} = 0$, it is also given in [84]. Preliminary results on symmetries for this integral were given in [91–93].

For uniform natural systems of degree -2, a special reduction can be made to lower the number of degrees of freedom by unity. We describe it in application to the considered system (102).

We carry out a substitution of time and a (projective) substitution of variables

$$dt = I\,d\tau, \quad \mathbf{G} = I^{-1/2}\mathbf{F}. \tag{107}$$

It can easily be shown by direct calculation that the evolution of the matrix $\mathbf{G}(t)$ can be described by a Lagrangian system with a constraint φ in the following form:

$$\begin{gathered} L = \frac{1}{2}\,\mathrm{Tr}\left(\frac{d\mathbf{G}}{d\tau}\frac{d\mathbf{G}^T}{d\tau}\right) - \bar{U}_g(\mathbf{G}), \quad \bar{U}_g(\mathbf{G}) = \frac{3}{2}k\frac{1}{(\det\mathbf{G})^{2/3}}, \\ \varphi = \mathrm{Tr}(\mathbf{G}\mathbf{G}^T) = 1. \end{gathered} \tag{108}$$

A relationship between the "old" time t and the "new" time τ can be found using (104). Note that the system (108) differs from the Dirichlet system, since the constraint φ is different in this case (in the Dirichlet problem, $\det\mathbf{G} = 1$). It is interesting that the energy integral for the system (108) coincides with the integral (106):

$$\bar{\mathcal{E}} = \frac{1}{4}J = \frac{1}{2}\,\mathrm{Tr}\left(\frac{d\mathbf{G}}{d\tau}\frac{d\mathbf{G}^T}{d\tau}\right) - \bar{U}_g(\mathbf{G}).$$

The linear integrals in the system (108) remains the same,

$$\mathbf{\Xi} = \mathbf{G}^T\frac{d\mathbf{G}}{d\tau} - \frac{d\mathbf{G}^T}{d\tau}\mathbf{G}, \quad \mathbf{M} = \mathbf{G}\frac{d\mathbf{G}^T}{d\tau} - \frac{d\mathbf{G}}{d\tau}\mathbf{G}^T;$$

furthermore, the system (108) is invariant with respect to the same transformations (97), which form a group $\Gamma = SO(3) \otimes SO(3)$. Therefore, a symmetry-based reduction similar to the above-described one is possible (see Part II, Section 3), with the only difference that, in this case, the following relationship between the semiaxes is valid:

$$\bar{A}_1^2 + \bar{A}_2^2 + \bar{A}_3^2 = 1. \tag{109}$$

We use the Riemann decomposition of the matrix $\mathbf{G} = \mathbf{Q}^T\bar{\mathbf{A}}\mathbf{\Theta}$, $\mathbf{Q}, \mathbf{\Theta} \in SO(3)$, $\bar{\mathbf{A}} = \mathrm{diag}(\bar{A}_1, \bar{A}_2, \bar{A}_3)$, to obtain, in this case, a system similar to (100) but with an additional constraint (109). To take this constraint into account and represent the equations in the most symmetric form, we define variables $\boldsymbol{q}$ and $\boldsymbol{K}$ according to the formulas

$$q_i = A_i, \quad \boldsymbol{K} = \boldsymbol{q} \times \frac{d\boldsymbol{q}}{d\tau}. \tag{110}$$

Then we finally obtain a reduced system in the form

$$\frac{d\boldsymbol{K}}{d\tau}=\boldsymbol{K}\times\frac{\partial\bar{H}}{\partial\boldsymbol{K}}+\boldsymbol{q}\times\frac{\partial\bar{H}}{\partial\boldsymbol{q}},\quad\frac{d\boldsymbol{q}}{d\tau}=\boldsymbol{q}\times\frac{\partial\bar{H}}{\partial\boldsymbol{K}},$$
$$\frac{d\boldsymbol{m}}{d\tau}=\boldsymbol{m}\times\frac{\partial\bar{H}}{\partial\boldsymbol{m}}.\quad\frac{d\boldsymbol{\mu}}{d\tau}=\boldsymbol{\mu}\times\frac{\partial\bar{H}}{\partial\boldsymbol{\mu}},\tag{111}$$
$$\bar{H}=\frac{1}{2}(\boldsymbol{K},\boldsymbol{K})+\frac{1}{4}\sum\left(\frac{m_i+\mu_i}{q_j-q_k}\right)^2+\left(\frac{m_i-\mu_i}{q_j+q_k}\right)^2+U_g(\boldsymbol{q}).$$

The (nonzero) Poisson brackets corresponding to the system (110) are as follows:

$$\{K_i,K_j\}=\varepsilon_{ijk}K_k,\quad\{K_i,q_j\}=\varepsilon_{ijk}q_k,\quad\{m_i,m_j\}=\varepsilon_{ijk}m_k,\quad\{\mu_i,\mu_j\}=\varepsilon_{ijk}\mu_k.$$

Thuis Poisson structure, as is known, corresponds to the algebra $e(3)\oplus so(3)\oplus so(3)$ and has four Casimir functions,

$$\Phi_K=(\boldsymbol{K},\boldsymbol{q}),\quad\Phi_q=(\boldsymbol{q},\boldsymbol{q}),$$
$$\Phi_m=(\boldsymbol{m},\boldsymbol{m}),\quad\Phi_\mu=(\boldsymbol{\mu},\boldsymbol{\mu});$$

in view of the definition, (110), of the variables $\boldsymbol{K}$ and $\boldsymbol{q}$, we have in this case

$$\Phi_K=0,\quad\Phi_q=1.$$

Thus, we ultimately conclude that

1. *if* $\Phi_m,\Phi_\mu\neq0$, *equations* (100) *correspond to a Hamiltonian system with four degrees of freedom;*

2. *if* $\Phi_m=0$ (*or* $\Phi_\mu=0$), *we obtain a system with three degrees of freedom;*

3. *if* $\Phi_m=\Phi_\mu=0$, *we obtain a system with two degrees of freedom.*

As already mentioned above, Gaffet [85] noted, for the case of $\Phi_\mu=0$, two additional first integrals (of the sixth degree in the velocities) independent of the energy integral and put forward the hypothesis of the integrability of the system in this case.

Moreover, it was stated in [85] that the system (102) at $\mathbf{\Xi}=0$ (or $\mathbf{M}=0$) is integrable in Liouville's sense; missing integrals are presented, although their commutativity is not shown. The missing integrals are polynomials of the sixth degree in momenta and have the form

$$I_6=36k^2\left(Y_0Y_2-\frac{1}{4}Y_1^2+3X_2+T(X_0+Y_0^2)\right)$$
$$+6k(4T^2Y_0+3PY_1+6TY_2)+27P^2+4T^3,$$
$$L_6=\left(\mathbf{A}^2m,\mathbf{V}_0\mathbf{A}^2m\times(\mathbf{V}_0^2\mathbf{A}^2m+\frac{3k}{(q_1q_2q_3)^{2/3}}m)\right),$$

where $\mathbf{A}=\operatorname{diag}(q_1,q_2,q_3)$ and the quantities X_i,Y_i,P, and T can be expressed in terms of the symmetric matrix

$$\mathbf{V}_0=\begin{pmatrix}\frac{1}{3}\sum\limits_{i=1}^{3}\frac{K_i}{q_i}-\frac{K_1}{q_1} & \frac{m_3}{q_1^2-q_2^2} & \frac{m_3}{q_3^2-q_1^3}\\ \frac{m_3}{q_1^2-q_2^2} & \frac{1}{3}\sum\limits_{i=1}^{3}\frac{K_i}{q_i}-\frac{K_2}{q_2} & \frac{m_1}{q_2^2-q_3^2}\\ \frac{m_2}{q_3^2-q_1^2} & \frac{m_1}{q_2^2-q_3^2} & \frac{1}{3}\sum\limits_{i=1}^{3}\frac{K_i}{q_i}-\frac{K_3}{q_3}\end{pmatrix}$$

as follows:

$$X_k = (q_1q_2q_3)^{2(k-1)/3}\,\mathrm{Tr}(\mathbf{V}_0^k\mathbf{A}^2), \qquad Y_k = (q_1q_2q_3)^{2(k+1)/3}\,\mathrm{Tr}(\mathbf{V}_0^k\mathbf{A}^{-2}),$$
$$T = -\frac{1}{2}(q_1q_2q_3)^{4/3}\,\mathrm{Tr}(\mathbf{V}_0^2), \qquad P = (q_1q_2q_3)^2\det\mathbf{V}_0.$$

In the case of $\Phi_\mu = 0$, the system (100) is an Euler–Calogero–Moser system of type D_3 [94, 95] with the potential U_g. The Lax representation of the given system without a potential can be found, e. g., in [96]. In the case where $U_g = \frac{3}{2}\frac{k}{(q_1q_2q_2)^{2/3}}$, equations (100) can be written as

$$\begin{cases} \dot{\mathbf{L}} = [\mathbf{L}, \mathbf{A}] + \dfrac{k}{(q_1q_2q_2)^{2/3}}\mathbf{D}^{-1}, \\ \dot{\mathbf{D}} = [\mathbf{D}, \mathbf{A}] + \mathbf{L}, \\ \dot{\mathbf{l}} = [\mathbf{l}, \mathbf{A}], \end{cases} \tag{112}$$

where the matrices

$$\mathbf{L} = \begin{pmatrix} p_1 & \frac{m_3}{q_2-q_1} & \frac{-m_2}{q_3-q_1} & \frac{-m_2}{q_3+q_1} & \frac{m_3}{q_2+q_1} & 0 \\ \frac{m_3}{q_2-q_1} & p_2 & \frac{m_1}{q_3-q_2} & \frac{m_1}{q_3+q_2} & 0 & \frac{-m_3}{q_2+q_1} \\ \frac{-m_2}{q_3-q_1} & \frac{m_1}{q_3-q_2} & p_3 & 0 & \frac{-m_1}{q_3+q_2} & \frac{m_2}{q_3+q_1} \\ \frac{-m_2}{q_3+q_1} & \frac{m_1}{q_3+q_2} & 0 & -p_3 & \frac{-m_1}{q_3-q_2} & \frac{m_2}{q_3-q_1} \\ \frac{m_3}{q_2+q_1} & 0 & \frac{-m_1}{q_3+q_2} & \frac{-m_1}{q_3-q_2} & -p_2 & \frac{-m_3}{q_2-q_1} \\ 0 & \frac{-m_3}{q_2+q_1} & \frac{m_2}{q_3+q_1} & \frac{m_2}{q_3-q_1} & \frac{-m_3}{q_2-q_1} & -p_1 \end{pmatrix},$$

$$\mathbf{l} = \begin{pmatrix} 0 & m_3 & -m_2 & m_2 & -m_3 & 0 \\ -m_3 & 0 & m_1 & -m_1 & 0 & m_3 \\ m_2 & -m_1 & 0 & 0 & m_1 & -m_2 \\ -m_2 & m_1 & 0 & 0 & -m_1 & m_2 \\ m_3 & 0 & -m_1 & m_1 & 0 & -m_3 \\ 0 & -m_3 & m_2 & -m_2 & m_3 & 0 \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} 0 & \frac{m_3}{(q_2-q_1)^2} & \frac{-m_2}{(q_3-q_1)^2} & \frac{m_2}{(q_3+q_1)^2} & \frac{-m_3}{(q_2+q_1)^2} & 0 \\ \frac{-m_3}{(q_2-q_1)^2} & 0 & \frac{m_1}{(q_3-q_2)^2} & \frac{-m_1}{(q_3+q_2)^2} & 0 & \frac{m_3}{(q_2+q_1)^2} \\ \frac{m_2}{(q_3-q_1)^2} & \frac{-m_1}{(q_3-q_2)^2} & 0 & 0 & \frac{m_1}{(q_3+q_2)^2} & \frac{-m_2}{(q_3+q_1)^2} \\ \frac{-m_2}{(q_3+q_1)^2} & \frac{m_1}{(q_3+q_2)^2} & 0 & 0 & \frac{-m_1}{(q_3-q_2)^2} & \frac{m_2}{(q_3-q_1)^2} \\ \frac{m_3}{(q_2+q_1)^2} & 0 & \frac{-m_1}{(q_3+q_2)^2} & \frac{m_1}{(q_3-q_2)^2} & 0 & \frac{-m_3}{(q_2-q_1)^2} \\ 0 & \frac{-m_3}{(q_2+q_1)^2} & \frac{m_2}{(q_3+q_1)^2} & \frac{-m_2}{(q_3-q_1)^2} & \frac{m_3}{(q_2-q_1)^2} & 0 \end{pmatrix}$$

form an $L - A$ pair for the system without a potential, and the matrix $\mathbf{D}$ has the form $\mathbf{D} = \mathrm{diag}(q_1, q_2, q_3, -q_3, -q_2, -q_1)$.

We failed to write the general equations (112) in the form of a normal $L - A$ pair. For the above-presented system with a third-degree integral, the $L - A$ pair was obtained in [97] using a completely different technique. Here, the question of generalizing the construction of this $L - A$ pair to the Gaffet system should be raised.

5.2. The Case of Axial Symmetry

As in the case of a fluid ellipsoid, it can easily be shown that the system (92) admits a three-dimensional invariant manifold formed by matrices of the form

$$F = \begin{pmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & w \end{pmatrix}. \tag{113}$$

The liner integrals (98) simplify in this case becoming

$$\Xi_{12} = -M_{12} = 2(u\dot{v} - v\dot{u}), \quad \Xi_{13} = \Xi_{23} = M_{13} = M_{23} = 0. \tag{114}$$

Consider the Ovsyannikov model with gravitation (93) and make the substitution of variables

$$u = \frac{1}{\sqrt{2}} r\cos\psi, \quad v = \frac{1}{\sqrt{2}} r\sin\psi, \quad w = z.$$

Then the Lagrangian function of the system assumes the form

$$\begin{gathered} L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\psi}^2 + \dot{z}^2) - U_g(r,z), \\ U_g = \frac{k}{\gamma - 1}\frac{1}{(r^2 z)^{\gamma-1}} + U_e(r,z), \end{gathered} \tag{115}$$

where the energy of the gravitational field U_e can be expressed in terms of elementary functions:

$$U_e = -2\varepsilon \int\limits_0^\infty \frac{d\lambda}{(\lambda + \frac{r^2}{2})\sqrt{\lambda + z^2}} = -\frac{2\varepsilon}{z} \times \begin{cases} \dfrac{2\,\mathrm{arctg}\sqrt{\chi^2 - 1}}{\sqrt{\chi^2 - 1}}, & \chi > 1, \\ \dfrac{\ln\left(\frac{1+\sqrt{1-\chi^2}}{1-\sqrt{1-\chi^2}}\right)}{\sqrt{1-\chi^2}}, & \chi < 1, \end{cases}$$

where $\chi = \frac{1}{\sqrt{2}}\frac{r}{z}$ is the semiaxis ratio. Since the Lagrangian (115) is independent of ψ, there is the cyclic integral

$$\frac{\partial L}{\partial \dot{\psi}} = r^2\dot{\psi} = c = \mathrm{const},$$

which coincides with the integrals (114) within a multiplier.

For a fixed value of this integral, we make the Legendre transformation $p_r = \frac{\partial L}{\partial \dot{r}} = \dot{r}, p_z = \frac{\partial L}{\partial \dot{z}} = \dot{z}$ and obtain a Hamiltonian system with two degrees of freedom in the canonical form

$$H = \frac{1}{2}(p_r^2 + p_z^2) + U_*(r,z), \quad U_* = \frac{c^2}{2r^2} + U_g(r,\psi); \tag{116}$$

here, U_* is the reduced potential.

Consider the simplest (integrable) case, the motion of a monoatomic gas ($\gamma = \frac{5}{3}$) without allowances for gravitation (i. e., $U_e = 0$; see also the preceding section).

It was shown above that the system in this case admits a reduction by one more degree of freedom and, therefore, reduces to a quadrature. Indeed, we make a substitution of variables and time of the form

$$r = R\cos\theta, \quad z = R\sin\theta, \quad dt = R^2\,d\tau,$$

where, in view of the conditions $r > 0$ and $z > 0$, the variable $\theta \in (0, \pi/2)$. We obtain the following equations for R and θ:

$$\frac{d^2}{dt^2}(R^2) = 4H = \text{const},$$
$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}\frac{c^2}{\cos^2\theta} + \frac{3}{2}\frac{k}{(\cos^2\theta\sin\theta)^{2/3}} = h_1 = \text{const}.$$

The quadrature for θ at $c = 0$, with certain limitations on the initial conditions, was obtained in [80]. As we can see, the evolution of $\theta(t)$ can be determined by the reduced potential

$$\bar{U}_*(\theta) = \frac{1}{2}\frac{c^2}{\cos^2\theta} + \frac{3}{2}\frac{k}{(\cos^2\theta\sin\theta)^{2/3}}.$$

At all values of the parameters c and k, this function has one critical value θ_0 in the interval $(0, \pi/2)$, in which $\bar{U}_*$ reaches its minimum. This value corresponds to the self-similar expansion of a spheroidal gas cloud. In other cases, the expansion of the cloud is accompanied by oscillations in the semiaxis lengths, with θ varying in the interval (θ_1, θ_2), where θ_i are the roots of the equation $\bar{U}_*(\theta) = h_1$.

In the general case, $U_e \neq 0$, the trajectories of the system (116) are not finite. However, it can easily be shown that, at $k > \frac{1}{96}2^{2/3}(\sqrt{665} - 21)c^2 \approx 0,43c^2$, the reduced potential has a minimum at the point

$$\theta_0 = \operatorname{arctg}\frac{1}{\sqrt{2}}, \quad R_0 = \frac{\sqrt{3}}{8\varepsilon}(c^2 + 3 \cdot 2^{1/3}k).$$

Therefore, near the minimum of the energy $U_*(\theta_0, R_0)$, the trajectories of the system are finite and a Poincaré map can be constructed. Such a map in the plane $\theta = \frac{\pi}{4}$ as the plane of section is shown in Fig. 3. A chaotic layer that originates from the splitting of resonant tori can be clearly seen in this figure, which testifies to the nonintegrability of the system (116).

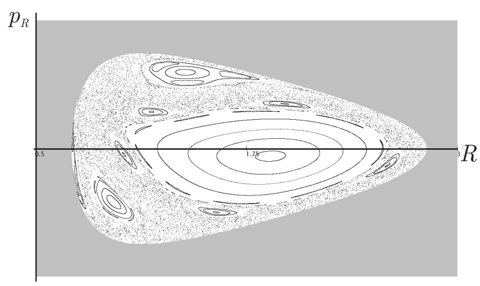

Fig. 3. The Poincaré map of the system (116) at $k = c = \varepsilon = 1$ in the section plane $\theta = \frac{\pi}{4}$.

5.3. Generalization of the Riemannian Case

An invariant manifold of the form (57) also exists for gas ellipsoids, i. e.,

$$\mathbf{F} = \begin{Vmatrix} u_1 & v_1 & 0 \\ u_2 & v_2 & 0 \\ 0 & 0 & w_3 \end{Vmatrix}. \tag{117}$$

As in the Riemannian case, it can be shown for a fluid ellipsoid that, in the case of gas, the following relationships are also valid:

$$m_1 = m_2 = \mu_1 = \mu_2 = 0, \quad m_3 = \text{const}, \quad \mu_3 = \text{const}.$$

Thus, we conclude that, according to (100), the evolution of the semiaxes $A_i = q_i, i = 1, 2, 3$ can be described by the third-degree Hamiltonian system

$$H = \frac{1}{2}\boldsymbol{p}^2 + U_*(q), \quad U_* = \frac{c_1^2}{(q_1 - q_2)^2} + \frac{c_2^2}{(q_1 + q_2)^2} + U_g(\boldsymbol{q}), \tag{118}$$

where $\boldsymbol{q}, \boldsymbol{p}$ are canonically conjugate variables and $c_1 = \frac{1}{2}(m_3 + \mu_3)$, $c_2 = \frac{1}{2}(m_3 - \mu_3)$ are fixed constants.

It was shown above that, for a monatomic gas ($\gamma = \frac{5}{3}$), without taking into account gravitation ($U_e = 0$), the system admits a reduction by one more degree of freedom. As a result, we obtain in this case a system of the form

$$\frac{d\boldsymbol{K}}{d\tau} = \boldsymbol{K} \times \frac{\partial \bar{H}}{\partial \boldsymbol{K}} + \boldsymbol{q} \times \frac{\partial \bar{H}}{\partial \boldsymbol{q}}, \quad \frac{d\boldsymbol{q}}{d\tau} = \boldsymbol{q} \times \frac{\partial \bar{H}}{\partial \boldsymbol{K}},$$

$$\bar{H} = \frac{1}{2}\boldsymbol{K}^2 + \bar{U}_*(\boldsymbol{q}), \quad \bar{U}_*(\boldsymbol{q}) = \frac{3}{2}\frac{k}{(q_1q_2q_3)^{2/3}} + \frac{c_1^2}{(q_1 - q_2)^2} + \frac{c_2^2}{(q_1 + q_2)^2}.$$

This system is equivalent to the problem of the motion of a spherical top in an axisymmetric potential [63]. As shown in [98, Sec. 4], this system is integrable provided $c_1^2 = c_2^2$. At $c_1 = c_2 = 0$, the additional integral of the third degree in the velocities has the form

$$F_3 = K_1K_2K_3 - 3k\frac{K_1q_2q_3 + K_2q_3q_1 + K_3q_1q_2}{(q_1q_2q_3)^{2/3}}.$$

If $c_1 = c_2 = c \neq 0$, we have an additional sixth-degree integral

$$F_6 = (F_3 + F_a)^2 + 4\frac{f(q_1^2\theta + 3kq_3^2)(q_2^2\theta + 3kq_3^2)}{q_3^4},$$

where

$$F_a = \frac{4c^2q_1q_2q_3^2}{(q_1^2 - q_2^2)^2}K_3, \quad f = \frac{4c^2(q_1q_2q_3)^{2/3}}{(q_1^2 - q_2^2)^2}q_3^2, \quad \theta = \frac{(q_1q_2q_3)^{2/3}}{q_1q_2}K_1K_2 - 3k + f.$$

In the more general case of $c_1^2 \neq c_2^2$, the system (117) becomes nonintegrable.

Figure 4 shows the corresponding Poincaré map in the Anduaye variables, which are traditionally used for reductions in the problems of rigid-body motion with a fixed point [63]. The break down of the resonant tori and the birth of isolated periodic solution can be clearly seen from the figure, which is evidence for the nonintegrability of the problem.

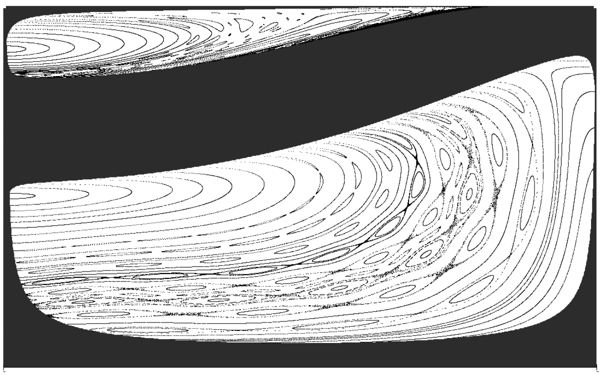

Fig. 4. Poincaré map of the system (117) at an energy level of $\bar{H} = 30$ at $k = 1/3, c_1 = 1, c_2 = 0.3$ in the section plane $g = \pi$.

6. ACKNOWLEDGMENTS

Authors thank B. Gaffet, V. V. Kozlov and F. Fassó for useful discussions. This work was supported by the Russian Foundation for Basic Researcvh (project codes 08-01-00651 and 07-01-92210). The work of I. S. Mamaev and A. A. Kilin was done in the framework of the program of the President of the Russian Federation for the support of young scientists, Doctors of Science (project code MD-5239.2008.1) and Candidates of Science (project code MK-6376.2008.1), respectively.

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