# Forms of equilibrium of liquid self-gravitating inhomogeneous mass 

A. Borisov, I. Mamaev

Udmurt State University

To date many results have been obtained about equilibrium forms of homogeneous liquid:

- the Maclaurin spheroid (1742);
- the Jacobi ellipsoid (1834);
- Poincaré and Liapounov obtained infinitely close to the above mentioned ellipsoid using Lamé's functions.
The problem of determining equilibrium forms of inhomogeneous bodies is mach more difficult and has been studied less.
- Clairaut (1743)
- Hamy (1887)
- Veronnet (1919)
- Pizzetty (1913)
- Montalvo D. Martinez F. J. Cisneros J. ( On equilibrium figures for ideal fluids in the form of confocal spheroids rotating with common and different angular velocities, 1982)
- Chaplygin (1948)

In our work we have obtained the joint solution of the hydrodynamic Euler equation, equation of continuity and the Poisson equation for gravity potential. The solution which we have obtained corresponds to the gomofocal spheroid, in which the angular velocity and density in each layer take on a constant value.
The spheroid surface is governed by the equation:

$$
\frac{x^{2}+y^{2}}{a^{2}}+\frac{z^{2}}{b^{2}}=1
$$

where $a>b$. The Euler equation that defines the motion of liquid is

$$
\begin{equation*}
\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V} \nabla) \mathbf{V}=\nabla U-\frac{1}{\rho} \nabla p \tag{1}
\end{equation*}
$$

The equation of continuity has the form:

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{V})=0 \tag{2}
\end{equation*}
$$

The potential $U$ satisfies the Poisson equation.

$$
\begin{equation*}
\Delta U=-4 \pi G \rho \tag{3}
\end{equation*}
$$

We will consider the system of equations in the curvilinear system $(r, \varphi, \mu)$ :

$$
\begin{gather*}
x=r \cos (\varphi), y=r \sin (\varphi), z=\left(b^{2}+\mu-\frac{b^{2}+\mu}{a^{2}+\mu} r^{2}\right)^{\frac{1}{2}},  \tag{4}\\
a \leq r \leq 0,2 \pi \leq \varphi \leq 0, \mu_{0} \leq \mu<-b^{2} . \tag{5}
\end{gather*}
$$

We consider the spheroid for which the field of velocities inside the spheroid has the form: $\omega=\omega(\mu)$, that is each layer which has the fixed value $\mu$ rotates with its constant angular velocity.
In this case the Euler equations take the following form in our system of coordinates:

$$
\begin{gather*}
-\omega(\mu)^{2} r=\frac{\partial U}{\partial r}-\frac{1}{\rho(\mu)} \frac{\partial p}{\partial r}+\frac{2\left(a^{2}+\mu\right)\left(b^{2}+\mu\right) r}{\left(a^{2}+\mu\right)^{2}-r^{2}\left(a^{2}-b^{2}\right)}\left(\frac{\partial U}{\partial \mu}-\frac{1}{\rho(\mu)} \frac{\partial p}{\partial \mu}\right)  \tag{6}\\
0=\frac{1}{r^{2}}\left(\frac{\partial U}{\partial \varphi}-\frac{\partial p}{\partial \varphi}\right)  \tag{7}\\
-\omega(\mu)^{2} r^{2}=r\left(\frac{\partial U}{\partial r}-\frac{1}{\rho(\mu)}\right)+2\left(a^{2}+\mu\right)\left(\frac{\partial U}{\partial \mu}-\frac{1}{\rho(\mu)} \frac{\partial p}{\partial \mu}\right) \tag{8}
\end{gather*}
$$

The gravity potential is determined by the Poisson equation which has the following form in our system of coordinates:

$$
\begin{align*}
\frac{1}{r} \frac{\partial U}{\partial r}+\frac{\partial^{2} U}{\partial r^{2}} & +\frac{1}{r^{2}} \frac{\partial^{2} U}{\partial \varphi}+\frac{2\left(a^{2}+\mu\right)}{\left(a^{2}+\mu\right)^{2}-r^{2}\left(a^{2}-b^{2}\right)}\left(\left(a^{2}+2 b^{2}+3 \mu\right) \frac{\partial U}{\partial \mu}+\right. \\
& \left.+2\left(b^{2}+\mu\right)\left(r \frac{\partial^{2} U}{\partial \mu \partial r}+\left(a^{2}+\mu\right) \frac{\partial^{2} U}{\partial^{2} \mu}\right)\right)=-4 \pi G \rho \tag{9}
\end{align*}
$$

Equations (6)-(9) have the solution $\omega(\mu)$ if and only if the gravity potential inside the spheroid has the following form:

$$
\begin{equation*}
U=r^{2} f_{i}(\mu)+g_{i}(\mu) \tag{10}
\end{equation*}
$$

Considering (10), we find:

$$
\begin{gather*}
\omega(\mu)^{2}=-\frac{\rho\left(\mu_{0}\right)}{\rho(\mu) r}\left(\frac{\partial U}{\partial r}\right)_{\mu=\mu_{0}}+\frac{1}{\rho(\mu) r} \int_{\mu}^{\mu_{0}} \rho^{\prime}(\mu) \frac{\partial U}{\partial r} d \mu,  \tag{11}\\
p=p_{0}-\rho\left(\mu_{0}\right) U_{\mu=\mu_{0}}+\rho(\mu) U+\int_{\mu}^{\mu_{0}} \rho^{\prime}(\mu) U d \mu \tag{12}
\end{gather*}
$$

Substituting the potential (10) in the Poisson equation and equating the coefficients with degrees $r$, we obtain a system of differential equations for the determination of $f(\mu)$ and $g(\mu)$ inside the spheroid

$$
\begin{align*}
& f_{i}^{\prime \prime}(\mu)+\frac{\left(a^{2}+6 b^{2}+7 \mu\right)}{2\left(a^{2}+\mu\right)\left(b^{2}+\mu\right)} f_{i}^{\prime}(\mu)-\frac{\left(a^{2}-b^{2}\right)}{\left(a^{2}+\mu\right)^{2}\left(b^{2}+\mu\right)} f_{i}(\mu)= \\
&=\frac{\pi G \rho(\mu)\left(a^{2}-b^{2}\right)}{\left(a^{2}+\mu\right)^{2}\left(b^{2}+\mu\right)},  \tag{13}\\
& g_{i}^{\prime \prime}(\mu)+\frac{\left(a^{2}+2 b^{2}+3 \mu\right)}{2\left(a^{2}+\mu\right)\left(b^{2}+\mu\right)} g_{i}^{\prime}(\mu)=-\frac{\pi G \rho(\mu)+f_{i}(\mu)}{\left(b^{2}+\mu\right)} . \tag{14}
\end{align*}
$$

and outside the spheroid $(\rho(\mu)=0)$.
The solution that we have obtained satisfies the well-known solution for the Maclaurin spheroid and our solution includes the solution for two layers each of which has a constant density.

We have considered the spheroid with continuous inhomogeneous density:

$$
\begin{equation*}
\rho(\mu, \lambda)=\frac{\rho\left(-2 \mu^{4}-5 b^{2} \mu^{3}-3 b^{4} \mu^{2}+2 \lambda^{2}\left(2 \lambda+3 b^{2}\right)\left(\lambda+b^{2}\right)\right)}{\lambda^{2}\left(3 b^{4}+5 \lambda b^{2}+2 \lambda^{2}\right)}, \tag{15}
\end{equation*}
$$

where $b$ characterizes the size of the core and $\lambda$ is the parameter of the shell.



Fig. 1: The density function (15)
( $b=2, \lambda=40$ )
Fig. 2: The function $\frac{\omega^{2}}{2 \pi G \rho}(\mu)$

