The Hamiltonian Dynamics of Self-gravitating Liquid and Gas Ellipsoids

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1. The Dirichlet equations

The equations of the dynamics of a homogeneous, incompressible, ideal fluid of unit density in a Lagrangian form are in the case of potential forces applied to the fluid as follows:

$$\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^T \ddot{\boldsymbol{x}} = -\frac{\partial (U+p)}{\partial \boldsymbol{a}}, \tag{1}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$ are the initial positions of the material points of the medium (the so-called Lagrangian coordinates),
$\boldsymbol{x}(\boldsymbol{a}, t)$ are the coordinates of the points of the medium at the time t (i. e., $\boldsymbol{x}(\boldsymbol{a}, 0) = a$),
$U(\boldsymbol{a}, t)$ is the density of the potential energy of the external forces, $p(\boldsymbol{a}, t)$ is the pressure,
$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} = \left\| \frac{\partial x_i}{\partial a_j} \right\|$ is the matrix of the partial derivatives.
These equations must be supplemented with the incompressibility condition, which can be written is the case at hand as

$$\det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right) = 1. \tag{2}$$

Thus, we obtain a system of partial differential equations in which four quantities, viz., x_1, x_2, x_3, and p, are unknown as the functions of the variables $\boldsymbol{a}$ and t. To determine them, except initial conditions ($\boldsymbol{x}(\boldsymbol{a}, 0) = \boldsymbol{a}$, $\dot{\boldsymbol{x}}(\boldsymbol{a}, 0) = \boldsymbol{v}_0(\boldsymbol{a})$), also boundary conditions must be specified; in our case, the latter reduce to the statement that the pressure has the same value independent of $\boldsymbol{a}$ everywhere on the free surface.

Dirichlet noted that, if the potential of the external forces $U(\boldsymbol{a}, t)$ is a homogeneous quadratic function of the Lagrangian coordinates, i. e.

$$U(\boldsymbol{a}, t) = U_0(t) + (\boldsymbol{a}, \mathbf{V}(t)\boldsymbol{a}), \tag{3}$$

where $U_0(t)$ is independent of $\boldsymbol{a}$ and $\mathbf{V}(t)$ is a symmetric matrix, then the equations of motion (1), (2) admit a partial solution

$$\boldsymbol{x}(\boldsymbol{a},t) = \mathbf{F}(t)\boldsymbol{a}, \quad \det \mathbf{F}(t) = 1. \tag{4}$$

Here, $\mathbf{F}(t)$ is a 3×3 matrix.
In this case, the boundary conditions will be satisfied provided that the fluid has initially an ellipsoidal shape,

$$(\boldsymbol{a}, \mathbf{A}_0^{-2}\boldsymbol{a}) \leqslant 1, \tag{5}$$

where $\mathbf{A}_0 = \mathrm{diag}(A_1^0, A_2^0, A_3^0)$ is the matrix of the initial semiaxes and the pressure has the form

$$p(\boldsymbol{a},t) = p_0(t) + \sigma(t)(1 - (\boldsymbol{a}, \mathbf{A}_0^{-2}\boldsymbol{a})). \tag{6}$$

We substitute (3), (4), and (6) into (1) and (2) to obtain equations for the matrix $\mathbf{F}(t)$ and the function $\sigma(t)$ in the form

$$\begin{aligned} &\mathbf{F}^T\ddot{\mathbf{F}} = -2\mathbf{V} - 2\sigma\mathbf{A}_0^{-2}, \\ &\det \mathbf{F} = 1. \end{aligned} \qquad \textit{(the Dirichlet equations)} \tag{7}$$

As Dirichlet showed, the system of ten equations (7) for ten unknown functions $F_{ij}(t), \sigma(t)$, $i,j = 1,2,3$, is compatible.

Obviously, the transformation (4) changes the original ellipsoid (5) into the ellipsoid specified by the quadratic form

$$(\boldsymbol{x}, (\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T)^{-1}\boldsymbol{x}) \leqslant 1. \tag{8}$$

1.1. Gravitational potential

Now, we determine the right-hand sides of equations (7) We use the known representation of the gravitational potential for the interior of the ellipsoid in the system of the principal axes

$$U(\zeta) = -\frac{3}{4}mG\int_0^\infty \frac{d\lambda}{\Delta(\lambda)}\left(1-\sum_i \frac{\zeta_i^2}{A_i^2+\lambda}\right), \quad \Delta^2(\lambda) = \prod_i (A_i^2+\lambda), \tag{9}$$

where G is the gravitational constant and $m = \frac{4}{3}\pi\rho A_1 A_2 A_3$ is the mass of the ellipsoid. It is now necessary to represent (9) in terms of the elements of the transformation matrix $\mathbf{F}$ and in the Lagrangian coordinates $\boldsymbol{a}$.

$$\begin{aligned} &\Delta^2(\lambda) = \det(\mathbf{A}^2+\lambda\mathbf{E}) = \det(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T+\lambda\mathbf{E}),\\ &\sum_i \frac{\zeta_i^2}{A_i^2+\lambda} = (\zeta,(\mathbf{A}^2+\lambda\mathbf{E})^{-1}\zeta) = (\boldsymbol{a},\mathbf{F}^T(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T+\lambda\mathbf{E})^{-1}\mathbf{F}\boldsymbol{a}). \end{aligned} \tag{10}$$

Thus, we find the following representation for the matrix $\mathbf{V}$ in the Dirichlet equations:

$$\mathbf{V} = \varepsilon\int_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T+\lambda\mathbf{E})}}\mathbf{F}^T(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T+\lambda\mathbf{E})^{-1}\mathbf{F}, \quad \varepsilon = \frac{3}{4}mG; \tag{11}$$

it can be shown by direct calculations (see [2]) that $\mathbf{V}$ depends on the elements of the matrix $\mathbf{F}$ only through symmetric combinations of the form $\Phi_{ij} = \sum_k F_{ik}F_{jk}$, which are the dot products of columns of the matrix $\mathbf{F}$.

1.2. The Roche Problem

By the Roche problem, according to Jeans's terminology [4] (see also [11, 1]) we mean the problem of the interaction of a deformable body (satellite) and a spherical rigid body which move along circular Keplerian orbits. Actually, in [19] Roche considered the motion of the liquid mass under the action of a gravitating center (the notion of *Roche zones* traces back to this work). More general problem, where the second body does not have a spherical symmetry (i.e. the motion of two arbitrary bodies with mass centers moving along circular orbits), is called the Darwin problem [11].

Let a self-gravitating fluid mass move in the field of a spherically symmetric rigid body and both of these bodies rotate about their common center of mass in circular orbits. We choose a (moving) coordinate system $Ox_1x_2x_3$ with its origin at the center of mass of the ellipsoid and direct the Ox_1 axis toward the common center and the Ox_3 axis normally to the plane of rotation (see Fig. 1).

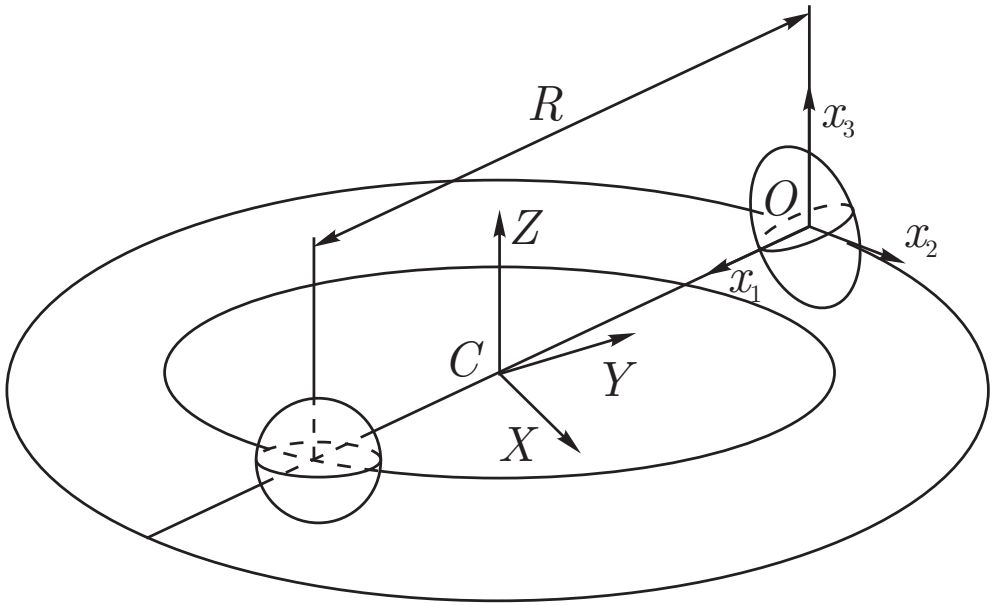

Fig. 1:

The equations of motion of incompressible fluid can be written in this case in the following Lagrangian form

$$\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^T (\ddot{\boldsymbol{x}} + 2\omega \boldsymbol{e}_3 \times \dot{\boldsymbol{x}}) = -\frac{\partial}{\partial \boldsymbol{a}} \left(p + U + U_s - \frac{1}{2}\omega^2(x_1^2 + x_2^2) + \omega^2 \frac{m_s}{m_e + m_s} R x_1\right), \tag{12}$$

$$\det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right) = 1, \tag{13}$$

where, as before, $\boldsymbol{a}$ are the Lagrangian coordinates of fluid elements, $\boldsymbol{x}(\boldsymbol{a}, t)$ are their positions at the given time, $p(\boldsymbol{a}, t)$ is the pressure, R is the distance between the centers of mass of the bodies, m_e and m_s are the masses of the ellipsoid and the sphere, respectively, ω is the angular velocity of rotation of the system about their common center of mass, and U is the gravitational potential (9). The gravitational potential of a spherical body U_s has the form

$$U_s = -\frac{m_s G}{\sqrt{(x_1 - R)^2 + x_2^2 + x_3^2}} = -\frac{m_s G}{R}\left(1 + \frac{x_1}{R} + \frac{1}{2}\frac{1}{R^2}\left(2x_1^2 - x_2^2 - x_3^2\right) + \ldots\right),$$

where G is the gravitational constant.
We omit higher-order terms in $\frac{|\boldsymbol{x}|}{R}$ and use the well-known relationship for a circular Keplerian orbit $R^3\omega^2 = G(m_e + m_s)$ to obtain finally (after collecting like terms) the equation

$$\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^T (\ddot{\boldsymbol{x}} + 2\omega \boldsymbol{e}_3 \times \dot{\boldsymbol{x}}) = -\frac{\partial}{\partial \boldsymbol{a}} \left(p + U - \frac{1}{2}\omega^2(\boldsymbol{x}, \mathbf{B}\boldsymbol{x})\right), \ \det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right) = 1, \tag{14}$$

where $\mathbf{B} = \mathrm{diag}\left(\frac{3m_s+m_e}{m_e+m_s}, \frac{m_e}{m_e+m_s}, -\frac{m_s}{m_e+m_s}\right)$. In the limiting case of a motionless Newtonian center $\left(\frac{m_e}{m_s} \to 0\right)$, we have $\mathbf{B} = \mathrm{diag}(3, 0, -1)$.
By substituting (4) into (6), we obtain the equations of motion in Roche's problem in the form

$$\mathbf{F}^T(\ddot{\mathbf{F}} + 2\Omega\dot{\mathbf{F}}) = -2\mathbf{V} + 2\sigma\mathbf{A}_0^{-2} + \omega^2\mathbf{F}^T\mathbf{B}\mathbf{F},$$
$$\det \mathbf{F} = 1, \tag{15}$$

where $\mathbf{\Omega} = \| - \omega\varepsilon_{ijk}\|$ is the matrix of the rotational velocity.

REMARK 1. Equations (14) are given in the book by Chandrasekhar, who uses them only to find hydrostatically equilibrated configurations of fluid masses and analyze their stability. Chandrasekhar does not present the dynamical equations (15).

2. First integrals

Let us return to the Dirichlet–Riemann problem on dynamics of the self-gravitating ellipsoid. The first integrals of the equations, linear in the velocities, can be obtained from the conservation laws for vorticity and angular momentum (the law of areas).

2.1. Vorticity

We write the law of conservation of vorticity for the hydrodynamic equations in the Lagrangian form (1), thus obtaining

$$\sum_i \left(\frac{\partial x_i}{\partial a_k} \frac{\partial \dot{x}_i}{\partial a_l} - \frac{\partial x_i}{\partial a_l} \frac{\partial \dot{x}_i}{\partial a_k} \right) = \xi_{kl} = \text{const}, \tag{16}$$

with the condition $\xi_{kl} = -\xi_{lk}$ satisfied. We denote this antisymmetric matrix as $\ddot{} = \|\xi_{kl}\|$ and find for the Dirichlet equations (7) that

$$\mathbf{\Xi} = \mathbf{F}^T \dot{\mathbf{F}} - \dot{\mathbf{F}}^T \mathbf{F} = \text{const}. \tag{17}$$

As already mentioned, the conservation of vorticity in this problem was noted by Dirichlet even before the appearance of a classical study by Helmholtz in which this law was extended to ideal hydrodynamics on the whole.

2.2. Momentum

The angular momentum relative to the center of the ellipsoid can be represented as

$$M_{ij} = \int (x_i \dot{x}_j - x_j \dot{x}_i) d^3 \boldsymbol{x} = \frac{m}{5} \sum_k (F_{ik} \dot{F}_{jk} - F_{jk} \dot{F}_{ik}) (A_k^0)^2. \tag{18}$$

In a matrix form, with the unimportant multiplier omitted, we have

$$\mathbf{M}' = \mathbf{F} \mathbf{A}_0^2 \dot{\mathbf{F}}^T - \dot{\mathbf{F}} \mathbf{A}_0^2 \mathbf{F}^T = \text{const}, \tag{19}$$

where $\mathbf{M}' = \|\frac{5}{m} M_{ij}\|$.

2.3. Energy

In addition to the linear integrals, the equations of motion also admit another, quadratic integral, viz., the total energy of the system. The integration of the kinetic and the potential energy of the fluid particles over the volume of the ellipsoid yields

$$\begin{aligned} \mathcal{E} &= \frac{m}{5}(T_e + U_e), \\ T_e &= \frac{1}{2}\,\mathrm{Tr}(\dot{\mathbf{F}}\mathbf{A}_0^2\dot{\mathbf{F}}^T) \\ U_e &= -2\varepsilon \int_0^\infty \frac{d\lambda}{\sqrt{(\lambda + A_1^2)(\lambda + A_2^2)(\lambda + A_3^2)}}. \end{aligned} \tag{20}$$

3. Lagrangian and Hamiltonian Formalism

It is known (see, e. g., [6]) that the motion of ideal fluid satisfies the Hamilton principle; therefore, Dirichlet's solution also satisfies this principle. This makes it possible to represent the equations of motion in a Lagrangian and, next, in a Hamiltonian form. The Hamiltonian principle for the considered problem was used for the first time by Lipschitz [8] and Padova [7].

As the Lagrangian function, it is necessary to choose the difference between the kinetic and potential energies of the fluid in the ellipsoid; within the unimportant multiplier, we have

$$L = T_e - U_e, \tag{21}$$

where T_e and U_e were defined above in (20). The elements of the matrix $\mathbf{F}$ appear as generalized coordinates. We write the Lagrange–Euler equations taking into account the constraint $\det \mathbf{F} = 1$ to obtain

$$\left(\frac{\partial L}{\partial \dot{\mathbf{F}}}\right)^{\cdot} - \frac{\partial L}{\partial \mathbf{F}} = \kappa \frac{\partial \varphi}{\partial \mathbf{F}}, \tag{22}$$

where $\varphi = \det \mathbf{F}$, and use the following matrix notation for any function: $\frac{\partial f}{\partial \mathbf{F}} = \left\| \frac{\partial f}{\partial F_{ij}} \right\|$, $\frac{\partial f}{\partial \dot{\mathbf{F}}} = \left\| \frac{\partial f}{\partial \dot{F}_{ij}} \right\|$, κ being the undefined Lagrangian multiplier. The differentiation in view of the formula $\left(\frac{\partial \varphi}{\partial \mathbf{F}}\right)^T = \varphi \mathbf{F}^{-1}$ yields

$$\ddot{\mathbf{F}}\mathbf{A}_0^2 = 2\varepsilon \frac{\partial}{\partial \mathbf{F}} \int\limits_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T + \lambda \mathbf{E})}} + \kappa (\mathbf{F}^{-1})^T \det \mathbf{F}. \tag{23}$$

We can easily make sure that these equations coincide with the Dirichlet equations (7) if we set $\kappa = 2\sigma$.

The matrix of the initial semiaxes $\mathbf{A}_0$ appears in the Lagrangian function and the equations of motion of the system as a set of parameters. Obviously, these parameters can be transferred to the initial conditions; indeed, upon the substitution $\mathbf{G} = \mathbf{F}\mathbf{A}_0$ (suggested by Dedekind [5]), the Lagrangian function and the equation of constraint can be written as

$$L = \frac{1}{2}\,\mathrm{Tr}(\dot{\mathbf{G}}\dot{\mathbf{G}}^T) + 2\varepsilon \int\limits_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{G}\mathbf{G}^T + \lambda\mathbf{E})}}, \tag{24}$$
$$\varphi = \det \mathbf{G} = \det \mathbf{A}_0 = \mathrm{const}.$$

The initial conditions have obviously the form $\mathbf{G}|_{t=0} = \mathbf{A}_0$, and the equation of motion preserves its form, $\left(\frac{\partial L}{\partial \dot{\mathbf{G}}}\right)^{\cdot} - \frac{\partial L}{\partial \mathbf{G}} = \widetilde{\mathcal{H}}\frac{\partial \varphi}{\partial G}$.
It can also be shown that the substitution

$$\mathbf{G} \to (\det \mathbf{A}_0)^{1/3}\mathbf{G}, \quad t \to \frac{(\det \mathbf{A}_0)^{1/3}}{2\varepsilon} t$$

reduces the system (24) to the case of $\varepsilon = 1/2, \varphi = 1$. Thus, the dynamics of the self-gravitating fluid ellipsoid is described by a natural Lagrangian system without parameters on the $SL(3)$ group.
The first integrals — vorticity, momentum (19), and energy (20) — can be represented in the form

$$\mathbf{\Xi} = \mathbf{G}^T\dot{\mathbf{G}} - \dot{\mathbf{G}}^T\mathbf{G}, \quad \mathbf{M} = \mathbf{G}\dot{\mathbf{G}}^T - \dot{\mathbf{G}}\mathbf{G}^T,$$
$$\mathcal{E} = \frac{1}{2}\,\mathrm{Tr}(\dot{\mathbf{G}}\dot{\mathbf{G}}^T) - 2\varepsilon \int\limits_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{G}\mathbf{G}^T + \lambda\mathbf{E})}}. \tag{25}$$

3.1. The Riemann Equations

Let us show how the equations of motion can be written in a Riemannian form. To this end, we pass to the moving system of the principal axes of the ellipsoid. It is known that such a transformation is given by the orthogonal matrix

$$\zeta = \mathbf{Q}\mathbf{x}, \quad \mathbf{Q}^T = \mathbf{Q}^{-1}. \tag{26}$$

In the new coordinates ζ, the ellipsoid is specified by the relationship

$$(\zeta, \mathbf{A}^{-2}\zeta) \leqslant 1, \tag{27}$$

where $\mathbf{A} = \mathrm{diag}(A_1, A_2, A_3)$ is the matrix of the principal semiaxes at the given time. We also note that, since the transform (4) is linear, the fluid particles constantly move over ellipsoids for which

$$(\zeta, \mathbf{A}^{-2}\zeta) = (\boldsymbol{a}, \mathbf{A}_0^{-2}\boldsymbol{a}) = n^2 = \mathrm{const}, \quad 0 \leqslant n^2 < 1. \tag{28}$$

(In particular, the fluid particles that were initially at the boundary remain at the boundary at any time). Therefore, the modulus of the vector $\mathbf{A}^{-1}\zeta$ does not vary, so that the vectors $\mathbf{A}^{-1}\zeta$ and $\mathbf{A}_0^{-1}\boldsymbol{a}$ are also related by the orthogonal transformation

$$\mathbf{A}^{-1}\zeta = \Theta\mathbf{A}_0^{-1}\boldsymbol{a}, \quad \Theta^T = \Theta^{-1}. \tag{29}$$

Thus, we obtain the following decomposition of the matrix $\mathbf{F}$:

$$\mathbf{F} = \mathbf{Q}^T\mathbf{A}\Theta\mathbf{A}_0^{-1}. \tag{30}$$

REMARK 2. Multiplying by a constant matrix $\mathbf{A}_0$ yields a decomposition of the form

$$\mathbf{F}\mathbf{A}_0 = \mathbf{Q}^T\mathbf{A}\Theta \tag{31}$$

known in linear algebra as a singular decomposition [15].

We introduce the angular velocities corresponding to the orthogonal transformations,

$$\mathbf{w} = \dot{\mathbf{Q}}\mathbf{Q}^T, \quad \boldsymbol{\omega} = \dot{\boldsymbol{\Theta}}\boldsymbol{\Theta}^T, \tag{32}$$

which are known to be antisymmetric matrices [18].

3.2. From here on, the components w_i and ω_i are related to the elements of the antisymmetric matrices (32) according to the regular rule

$$\mathrm{w}_{ij} = \varepsilon_{ijk}\mathrm{w}_k, \quad \omega_{ij} = \varepsilon_{ijk}\omega_k. \tag{33}$$

3.3. Thus the decomposition (30) represents the equations of motion on the configuration space $\mathbb{R}^2 \otimes \mathrm{SO}(3) \otimes \mathrm{SO}(3)$ (the direct product of the Abel group of translations and two copies of the group of rotations of three-dimensional space), with the elements of the matrices $\mathbf{w}$ and $\boldsymbol{\omega}$ corresponding to the velocity components with respect to the basis of left-invariant vector fields. The equations of motion assume the form of the Poincaré equations on the Lie group [18]; in view of the fact that the Lagrangian function (24) is independent of the elements of the matrices $\mathbf{Q}$ and $\boldsymbol{\Theta}$ and with due account for the constraint $\varphi = A_1A_2A_3 = \mathrm{const}$, we obtain the following representation of the Riemann equations:

$$\begin{gathered}\left(\frac{\partial L}{\partial \dot{A}_i}\right)^{\cdot} = \frac{\partial L}{\partial A_i} + \tilde{\kappa}\frac{\partial \varphi}{\partial A_i}, \\ \left(\frac{\partial L}{\partial \mathrm{w}_i}\right)^{\cdot} = \sum_{j,k}\varepsilon_{ijk}\frac{\partial L}{\partial \mathrm{w}_j}\mathrm{w}_k, \quad \left(\frac{\partial L}{\partial \omega_i}\right)^{\cdot} = \sum_{j,k}\varepsilon_{ijk}\frac{\partial L}{\partial \omega_j}\omega_k.\end{gathered} \tag{34}$$

where $\tilde{\kappa}$ is the Lagrangian undetermined multiplier (which coincides with σ within a multiplier) and ε_{ijk} is the Levi-Civita antisymmetric tensor.

3.4. The Riemann equations can be written in the following matrix form:

$$\begin{aligned} &\dot{\mathbf{v}} - \mathbf{w}\mathbf{v} + \mathbf{v}\boldsymbol{\omega} = -2\hat{\mathbf{V}}\mathbf{A} + 2\sigma\mathbf{A}^{-1}, \\ &\mathbf{v} = \dot{\mathbf{A}} - \mathbf{w}\mathbf{A} + \mathbf{A}\boldsymbol{\omega}, \qquad \textit{(the Riemann equations)} \\ &A_1A_2A_3 = 1, \end{aligned} \tag{35}$$

3.5. Where $\hat{\mathbf{V}} = \mathrm{diag}(\hat{V}_1, \hat{V}_2, \hat{V}_3)$, and

$$\hat{V}_i = \varepsilon \int_0^\infty \frac{1}{\lambda + A_i^2} \frac{d\lambda}{\Delta(\lambda)} = -\frac{1}{A_i}\frac{\partial}{\partial A_i}\varepsilon \int_0^\infty \frac{d\lambda}{\Delta(\lambda)}. \tag{36}$$

3.6. Symmetry Group and the Dedekind Reciprocity Law

The Lagrangian representation of the Dirichlet equations (22) offers a very simple way to finding the symmetry group of the system. Indeed, it can be shown that the Lagrangian with the constraint [see (24)] and, therefore, the equations of motion are invariant with respect to transformations of the form

$$\mathbf{G}' = \mathbf{S}_1\mathbf{G}\mathbf{S}_2, \quad \mathbf{S}_1, \mathbf{S}_2 \in SO(3). \tag{37}$$

Thus, the system is invariant with respect to the group $\Gamma = SO(3) \otimes SO(3)$.

Clearly, the Noether integrals corresponding to the transformations (37) are the integrals of vorticity and total momentum (25). Accordingly, as will be shown below, the Riemann equations describe a system reduced based on the given symmetry group.

Furthermore, it can easily be shown using (24) that the equations of motion are invariant with respect to the discrete transformation of transposition of matrices:

$$\mathbf{G}' = \mathbf{G}^T.$$

Therefore, we have

Theorem 1 (The Dedekind reciprocity law). *Any solution,* $\mathbf{G}(t)$*, of the Dirichlet equations can be placed in correspondence with the solution* $\mathbf{G}'(t) = \mathbf{G}^T(t)$ *for which the rotation of the ellipsoid and the rotation of the fluid inside the ellipsoid* (*i. e.,* $\boldsymbol{\Theta}$ *and* $\mathbf{Q}$*; see* (30)) *are interchanged.*

The most widely known example is the Dedekind ellipsoid reciprocal to the Jacobi ellipsoid. In this case, the axes of the three-axial ellipsoid are spatially invariable and the fluid inside it moves around the minor axis in closed ellipses [5, 3].

3.7. Hamiltonian Formalism and Symmetry-based Reduction

We represent the Riemann equations in a Hamiltonian form. To this end, we first use the constraint equation $\varphi = \mathrm{const}$ to find a representation of one semiaxis,

$$A_3 = \frac{v_0}{A_1 A_2}, \tag{38}$$

where v_0 is the volume of the ellipsoid (within a multiplier). We carry out the Legendre transformation

$$\begin{gathered} p_i = \frac{\partial L}{\partial \dot{A}_i}, \quad m_k = \frac{\partial L}{\partial \mathrm{w}_k}, \quad \mu_k = \frac{\partial L}{\partial \omega_k}, \quad i = 1, 2, \quad k = 1, 2, 3, \\ H = \sum_i p_i \dot{A}_i + \sum_k (m_k \mathrm{w}_k + \mu_k \omega_k) - L \,|_{\dot{A}, \omega, \mathrm{w} \to p, m, \mu}\,. \end{gathered} \tag{39}$$

It can be shown using the expressions for the integrals, that the vectors $\boldsymbol{m} = (m_1, m_2, m_3)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ are related to the momentum and vorticity of the ellipsoid via the formulas

$$\boldsymbol{m} = \mathbf{Q}^T \boldsymbol{M}', \qquad \boldsymbol{\mu} = \boldsymbol{\Theta}^T \boldsymbol{\xi}', \tag{40}$$

where the vectors $\boldsymbol{M}'$ and $\boldsymbol{\xi}'$ are constituted by the components of the antisymmetric matrices $\mathbf{M}'$ and $\boldsymbol{\Xi}'$ according to the normal rule (33). In the new variables, the equations of motion assume the form

$$\begin{gathered}\dot{A}_i = \frac{\partial H}{\partial \dot{p}_i}, \quad \dot{p}_i = \frac{\partial H}{\partial A_i}, \quad i = 1, 2, \\ \dot{\boldsymbol{m}} = \boldsymbol{m} \times \frac{\partial H}{\partial \boldsymbol{m}}, \quad \dot{\boldsymbol{\mu}} = \boldsymbol{\mu} \times \frac{\partial H}{\partial \boldsymbol{\mu}}.\end{gathered} \tag{41}$$

Here, the Hamiltonian is

$$\begin{gathered}H = H_A + H_{m\mu} + U_e, \\ H_A = \frac{1}{2} \frac{A_3^{-2}(p_1^2 + p_2^2) + (p_1 A_2^{-1} - p_2 A_1^{-1})^2}{\sum A_i^{-2}}, \\ H_{m\mu} = \frac{1}{4} \sum_{\text{cycle}} \left(\frac{m_i + \mu_i}{A_j - A_k} \right)^2 + \left(\frac{m_i - \mu_i}{A_j + A_k} \right)^2,\end{gathered} \tag{42}$$

where U_e is specified by formula (20) and it is assumed that A_3 is defined according to (38).

In addition, equations (41) must necessarily be supplemented with equations describing the evolution of the matrices $\mathbf{Q}$ and $\boldsymbol{\Theta}$; they have the form

$$\dot{Q}_{ij} = \sum_{k,l} \varepsilon_{ikl} Q_{kj} \frac{\partial H}{\partial m_l}, \quad \dot{\Theta}_{ij} = \sum_{k,l} \varepsilon_{ikl} \Theta_{kj} \frac{\partial H}{\partial \mu_l}. \tag{43}$$

Equations (41) and (43) form a Hamiltonian system with eight degrees of freedom and uncanonical Poisson brackets,

$$\{A_i, p_j\} = \delta_{ij}, \quad \{m_i, m_j\} = \varepsilon_{ijk} m_k, \quad \{\mu_i, \mu_j\} = \varepsilon_{ijk} \mu_k, \tag{44}$$

$$\{m_r, Q_{jk}\} = \varepsilon_{ikl} Q_{jl}, \quad \{\mu_i, \Theta_{jk}\} = \varepsilon_{ikl} \Theta_{jl}, \tag{45}$$

where zero brackets are omitted.

REMARK 3. The elimination of one semiaxis (38) results in the loss of symmetry of the Hamiltonian (42); therefore, the equations for the semiaxes A_i are normally left in the Lagrangian form with an undetermined multiplier [3, 1].

It can be seen from the above relationships that the system of equations (41), which describes the evolution of the variables $A_i, p_i, \boldsymbol{m}$, and $\boldsymbol{\mu}$, separates; in addition, the Poisson bracket of these variables, (44), also proves to be closed. It is not difficult to show that that equations (41) describe a system reduced over the symmetry group (37).
Limitation: the brackets (44) obviously have two Casimir functions,

$$\Phi_m = (\boldsymbol{m}, \boldsymbol{m}), \quad \Phi_\mu = (\boldsymbol{\mu}, \boldsymbol{\mu}), \tag{46}$$

and have a rank of eight (provided that $\Phi_m \neq 0, \Phi_\mu \neq 0$).
Therefore, the reduced system has generally four degrees of freedom.
In particular cases where *one of the integrals* (46) *is zero, the reduced system has three degrees of freedom*. These are so-called irrotational ($\Phi_\mu = 0$) and momentum-free ($\Phi_m = 0$) ellipsoids.
If both of the integrals (46) *vanish, the reduced system has two degrees of freedom* and describes oscillations of the ellipsoid without changes in the directions of the axes and without inner flows (this case will be considered below in detail).

REMARK 4. The canonical variables in the Riemann equations were introduced for the first time by Betti [9], who used the commutation representations of the $so(4)$ algebra long before the advent of the modern theory of Hamiltonian systems on the Lie algebras. With the use of commutation, he introduced, in a quite modern way, canonical variables to reduce the integration of the Riemann equations to the integration of the Hamilton–Jacobi equations. The Hamiltonian nature of the Riemann equations is also considered in modern studies [12, 13, 14], which are related to the representation of the equations of motion on an extended Lie algebra for which the actual motions are in special orbits; the value of such a calculation for dynamics is not yet clear to us. A more formal procedure of reduction and Hamiltonization of the Riemann equations nearly relevant to our study is described in [15]. An akin analysis is done in [16] in the context of the Dirichlet motions in ideal magnetohydrodynamics. An alternative approach to the Hamiltonian nature, which also should be discussed, is presented in [17].

4. Particular cases of motion

4.1. Shape-preserving Motions of the Ellipsoin

The simplest motions of the fluid ellipsoids are represented by a family of solutions for which all the three axes of the ellipsoid are time-independent,

$$A_i = \mathrm{const}, \quad i = 1, 2, 3. \tag{47}$$

Clearly, the Maclaurin and Jacobi ellipsoids are examples of such motions. In these cases, the ellipsoid rotates as a rigid body about the principal axis (the symmetry axis for the Maclaurin ellipsoid and the shortest axis for the Jacobi ellipsoid).

The Dedekind ellipsoid offers another example of such motions, the axes being invariable in both their lengths and directions. As noted above, the Dedekind ellipsoid is reciprocal to the Jacobi ellipsoid in terms of Theorem 1 (while the Maclaurin ellipsoid is self-reciprocal).

For all the above-mentioned solutions (the Maclaurin, Jacobi, and Dedekind ellipsoids), two pairs of components of the vectors $\boldsymbol{m}$ and $\boldsymbol{\mu}$ vanish, the remaining components being constant (for example, it can be assumed without loss of generality that $m_1 = \mu_1 = m_2 = \mu_2 = 0, m_3 = \mathrm{const}, \mu_3 = \mathrm{const}$).

Riemann [3] has proved the following, more general result:

Theorem 2. *Let* (47) *be satisfied and let all the* A_i *be different. Then* $\boldsymbol{m}$ *and* $\boldsymbol{\mu}$ *are time-independent and at least one pair of components of these vectors vanishes* (*i. e.* $m_i = \mu_i = 0$ *for some* i).

As a consequence, we find that any motion of a shape-preserving fluid ellipsoid whose axes do not coincide, is a fixed point of the reduced system (41) or, which is the same, of the Riemann equations. Another proof of this statement is given in [7].

Riemann also noted new solutions — the Riemann ellipsoids — for the case where only one pair of components of $\boldsymbol{m}$ and $\boldsymbol{\mu}$ vanishes (i. e. $m_1 = \mu_1 = 0, \mu_2, m_2, \mu_3, m_3 \neq 0$).

V. A. Stekloff [20, 21] analyzed in detail the case of equality of a pair of axes ($A_i = A_j \neq A_k$) and showed that no shape-preserving motions other than Maclaurin ellipsoids (spheroids) exist in this case. In this sense, he generalized the Riemann result to the axisymmetric case (Riemann himself gave no detailed proof for this case). An attempt of revising Riemann's results was made in [22].

4.2. Axisymmetric Case (Dirichlet [2])

It can easily be shown that the equations of motion determined by the Lagrangian function (24) admit a (two-dimensional) invariant manifold that consists of matrices of the form

$$\mathbf{G} = \begin{Vmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & w \end{Vmatrix},$$

where $\det \mathbf{G} = (u^2 + v^2)w = v_0 = \mathrm{const}$ is the volume of the ellipsoid. This manifold corresponds to an axisymmetric motion of the fluid ellipsoid (see [2]). In this case, the matrix of the principal semiaxes is

$$\mathbf{A} = (\mathbf{G}\mathbf{G}^T)^{1/2} = \mathrm{diag}(\sqrt{u^2 + v^2}, \sqrt{u^2 + v^2}, w).$$

In view of the condition $\det \mathbf{G} = v_0$, we make the substitution of variables

$$u = v_0^{1/3} r \cos\psi, \quad v = v_0^{1/3} r \sin\psi, \quad w = \frac{v_0^{1/3}}{r^2}$$

and find that the Lagrangian function (24) is

$$L = v_0^{2/3} \left(\left(1 + \frac{2}{r^6} \right) \dot{r}^2 + r^2 \dot{\psi}^2 + U_s \right),$$

where

$$U_s = -\frac{2\varepsilon}{v_0}\int\limits_0^\infty \frac{d\lambda}{(\lambda + r^2)\sqrt{\lambda + 1/r^4}} = -\frac{2\varepsilon}{v_0} r^2 \times \begin{cases} \dfrac{2\operatorname{arctg}\sqrt{r^6-1}}{\sqrt{r^6-1}}, & r > 1, \\ \dfrac{\ln\left(\frac{1+\sqrt{1-r^6}}{1-\sqrt{1-r^6}}\right)}{\sqrt{1-r^6}}, & r < 1. \end{cases}$$

The variable ψ is cyclic; therefore, we have a first integral of the form

$$p_\psi = \frac{1}{v_0^{2/3}}\frac{\partial L}{\partial \dot\psi} = 2r^2\dot\psi,$$

which coincides within a multiplier with the single nonzero component of the momentum M'_{12} (19). With the use of the energy integral (20), we obtain a quadrature that specifies the evolution of r:

$$\left(1 + \frac{2}{r^6}\right)\dot r^2 = h - U_*, \quad U_* = U_s + \frac{c}{r^2},$$

where $h = \frac{\mathcal{E}}{m v_0^{2/3}}$ and $c = \frac{p_\psi}{4}$ are fixed values of the energy and momentum integrals. The minimum of the reduced potential U_* corresponds to the Maclaurin spheroid.

4.3. Riemannian Case [3]

There is an invariant manifold more general than the above-described one. It is specified by the block–diagonal matrix of the general form

$$\mathbf{G} = \begin{Vmatrix} u_1 & v_1 & 0 \\ u_2 & v_2 & 0 \\ 0 & 0 & w_3 \end{Vmatrix}. \tag{48}$$

We compute the integrals (19) obtaining

$$M'_{12} = u_1\dot{u}_2 - u_2\dot{u}_1 + v_1\dot{v}_2 - v_2\dot{v}_1, \quad M'_{23} = M'_{13} = 0,$$
$$\xi'_{12} = u_1\dot{v}_1 - v_1\dot{u}_1 + u_2\dot{v}_2 - v_2\dot{u}_2, \quad \xi'_{23} = \xi'_{13} = 0,$$

It is also obvious that i$\mathbf{Q}$ and $\mathbf{\Theta}$ have in this case a block–diagonal form similar to (48); therefore, this case corresponds to that noted by Riemann, for which, in equations (41), we should set

$$m_1 = m_2 = 0, \qquad m_3 = \mathrm{const},$$
$$\mu_1 = \mu_2 = 0, \qquad \mu_3 = \mathrm{const}.$$

Thus, we obtain a Hamiltonian system with two degrees of freedom, which describes the evolution of the principal semiaxes A_1 and A_2; its Hamiltonian is

$$H = \frac{1}{2}\frac{A_3^{-2}(p_1^2 + p_2^2) + (p_1A_2^{-1} - p_2A_1^{-1})^2}{\sum A_i^{-2}} + U_*(A_1, A_2), \tag{49}$$

where the reduced potential is

$$U_* = U_e + \frac{c_1^2}{(A_1 - A_2)^2} + \frac{c_2^2}{(A_1 + A_2)^2},$$

and $c_1^2 = \frac{1}{4}(m_3 + \mu_3)^2$, $c_2^2 = \frac{1}{4}(m_3 - \mu_3)^2$ are fixed constants of the integrals.
The particular version of the system (49) for $c_1 = c_2 = 0$ (i. e., for invariable directions of the principal axes of the ellipsoid) was also noted by Kirchhoff [6], who suggested that the problem does not reduce to quadratures.

At $U_* = 0$, the Hamiltonian (49) describes a geodesic flow on the cubic $A_1A_2A_3 = \mathrm{const}$. This remarkable analogy between two different dynamical systems was also noted by Riemann.

4.3. Elliptic Cylinder (Lipschitz [8])

This case can be obtained through a limiting process in the Riemannian case, with one axis of the ellipsoid going to infinity ($A_3 \to \infty$). It is, however, more convenient to start with considering the case of a two-dimensional motion of fluid assuming that the matrix $\mathbf{F}$ has the form

$$\mathbf{F} = \left\| \begin{array}{c|c} \bar{\mathbf{F}} & 0 \\ \hline 0 & 1 \end{array} \right\|, \qquad \det \bar{\mathbf{F}} = 1, \tag{50}$$

where $\bar{\mathbf{F}}$ is a 2×2 matrix with unit determinant.

Obviously, the considerations on which the derivation of the Dirichlet equations [10] was based can be applied to this case without modifications; only the right-hand side of the equations should be properly changed. To this end, it is necessary to use the well-known representation of the potential of the interior points of the elliptic cylinder with a large length l in the system of principal axes

$$U(\zeta) = \bar{\varepsilon} \left(U_0(l) - \frac{\zeta_1^2}{A_1(A_1 + A_2)} - \frac{\zeta_2^2}{A_2(A_1 + A_2)} \right) + O(1/l),$$

where $\bar{\varepsilon} = G\bar{m}$, G is the gravitational constant and $\bar{m} = \pi\rho A_1 A_2$ is the mass per unit length of the cylinder. The constant $U_0(l) \underset{l \to \infty}{\longrightarrow} \infty$ does not appear in the equations of motion and can be omitted.

By analogy with the above considerations, we pass to the Lagrangian representation and make the substitution $\bar{\mathbf{G}} = \bar{\mathbf{F}}\bar{\mathbf{A}}_0$, where $\bar{\mathbf{A}}_0 = \mathrm{diag}(A_1^0, A_2^0)$, to obtain the Lagrangian of the system in the form

$$L = \frac{1}{2} \operatorname{Tr} \left(\dot{\bar{\mathbf{G}}} \dot{\bar{\mathbf{G}}}^T \right) - \bar{U}_e,$$

$$\bar{U}_e = -2\bar{\varepsilon} \ln(A_1 + A_2)^2 = -2\bar{\varepsilon} \ln(\operatorname{Tr}(\bar{\mathbf{G}}\bar{\mathbf{G}}^T) + 2 \det \bar{\mathbf{G}}).$$

Based on the singular decomposition of the matrix $\bar{\mathbf{G}} = \bar{\mathbf{Q}}^T \bar{\mathbf{A}} \bar{\mathbf{\Theta}}$ with

$$\bar{\mathbf{Q}} = \begin{Vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{Vmatrix}, \quad \bar{\mathbf{\Theta}} = \begin{Vmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{Vmatrix}, \quad \mathbf{A} = \begin{Vmatrix} A_1 & 0 \\ 0 & A_2 \end{Vmatrix},$$

explicitly substituted, we obtain a Lagrangian function in the form

$$L = \frac{1}{2}\left(\dot{A}_1^2 + \dot{A}_2^2 + (A_1\dot{\phi} - A_2\dot{\psi})^2 + (A_2\dot{\phi} - A_1\dot{\psi})^2\right) - \bar{U}_e(A_1, A_2).$$

We can see that the variables ϕ and ψ are cyclic, and there are two linear integrals

$$\frac{\partial L}{\partial \dot{\phi}} = p_\phi, \quad \frac{\partial L}{\partial \dot{\psi}} = p_\psi. \tag{51}$$

We parametrize the relationship $A_1 A_2 = \bar{v}_0$ using hyperbolic functions,

$$A_1 = \bar{v}_0^{1/2}(\operatorname{ch} u + \operatorname{sh} u), \quad A_2 = \bar{v}_0^{1/2}(\operatorname{ch} u - \operatorname{sh} u).$$

We use the energy integral and the integrals (51) to obtain a quadrature for the variable u:

$$\bar{v}_0(\operatorname{ch} 2u)\dot{u}^2 = h - \bar{U}_*,$$
$$\bar{U}_* = 2\bar{\varepsilon}\ln(\operatorname{ch} u) + \frac{\bar{c}_1^2}{\operatorname{ch}^2 u} + \frac{\bar{c}_2^2}{\operatorname{sh}^2 u},$$

where $\bar{c}_1^2 = \frac{1}{16}(p_\phi - p_\psi)^2$, $\bar{c}_2^2 = \frac{1}{16}(p_\phi + p_\psi)^2$, and h are fixed constants of the first integrals.

5. Chaotic oscillations of a three-axial ellipsoid

Let us consider in more detail the oscillations (pulsations) of a fluid ellipsoid in the Riemannian case (48). We now represent the equations of motion of the system (49) in a Hamiltonian form most convenient for a numerical investigation of the system. We parametrize the surface $A_1A_2A_3 = v_0$ using cylindrical coordinates

$$\begin{aligned} &A_1 = r\cos\phi, \quad A_2 = r\sin\phi, \quad A_3 = \frac{2v_0}{r^2\sin^2 2\phi}, \\ &p_1 = p_r\cos\phi - \frac{p_\phi}{r}\sin\phi, \quad p_2 = p_r\sin\phi - \frac{p_\phi}{r}\cos\phi \end{aligned} . \tag{52}$$

The Hamiltonian (49) can be represented in the form

$$H = \frac{1}{2}\left(1 + \frac{c_0^2}{r^6\sin^4 2\phi}\right)^{-1}\left(p_r^2 + \frac{p_\phi^2}{r^2} + \frac{c_0^2}{r^6\sin^4 2\phi}\left(p_r\cos 2\phi - \frac{p_\phi}{r}\sin 2\phi\right)^2\right) + U_*(r,\phi), \tag{53}$$

where $c_0 = 4v_0$.

Since the original system is defined in the quadrant $A_1 > 0, A_2 > 0, A_3 > 0$, for this case we have $0 < \phi < \pi/2$. In this system, the transformation of variables

$$\rho = r^2, \quad \psi = 2\phi, \tag{54}$$

enables obtaining the Hamiltonian in the form

$$H = \frac{2(\rho^2(c_0^2\cos^2\psi + \rho^3\sin^4\psi)p_\rho^2 + \sin^2\psi(c_0^2 + \rho^3\sin^2\psi)p_\phi^2 - 2\rho c_0^2\cos\psi\sin\psi p_\psi p_\phi)}{\rho(c_0^2 + \rho^3\sin^4\psi)} + \\ + U_*(\rho,\psi). \tag{55}$$

Upon passing to new Cartesian coordinates according to the formulas

$$x = \rho \cos \psi, \quad y = \rho \sin \psi, \tag{56}$$

we obtain

$$H = 2\rho \left(p_x^2 + \frac{y^4 p_y^2}{y^4 + \bar{c}_0^2 \rho} \right) + U_*(x, y), \tag{57}$$

where $\rho = \sqrt{x^2 + y^2}$; obviously, the system (57) is defined in the upper semiplane ($y > 0$). In this case, as we can see, the kinetic energy of the system has the simplest form.

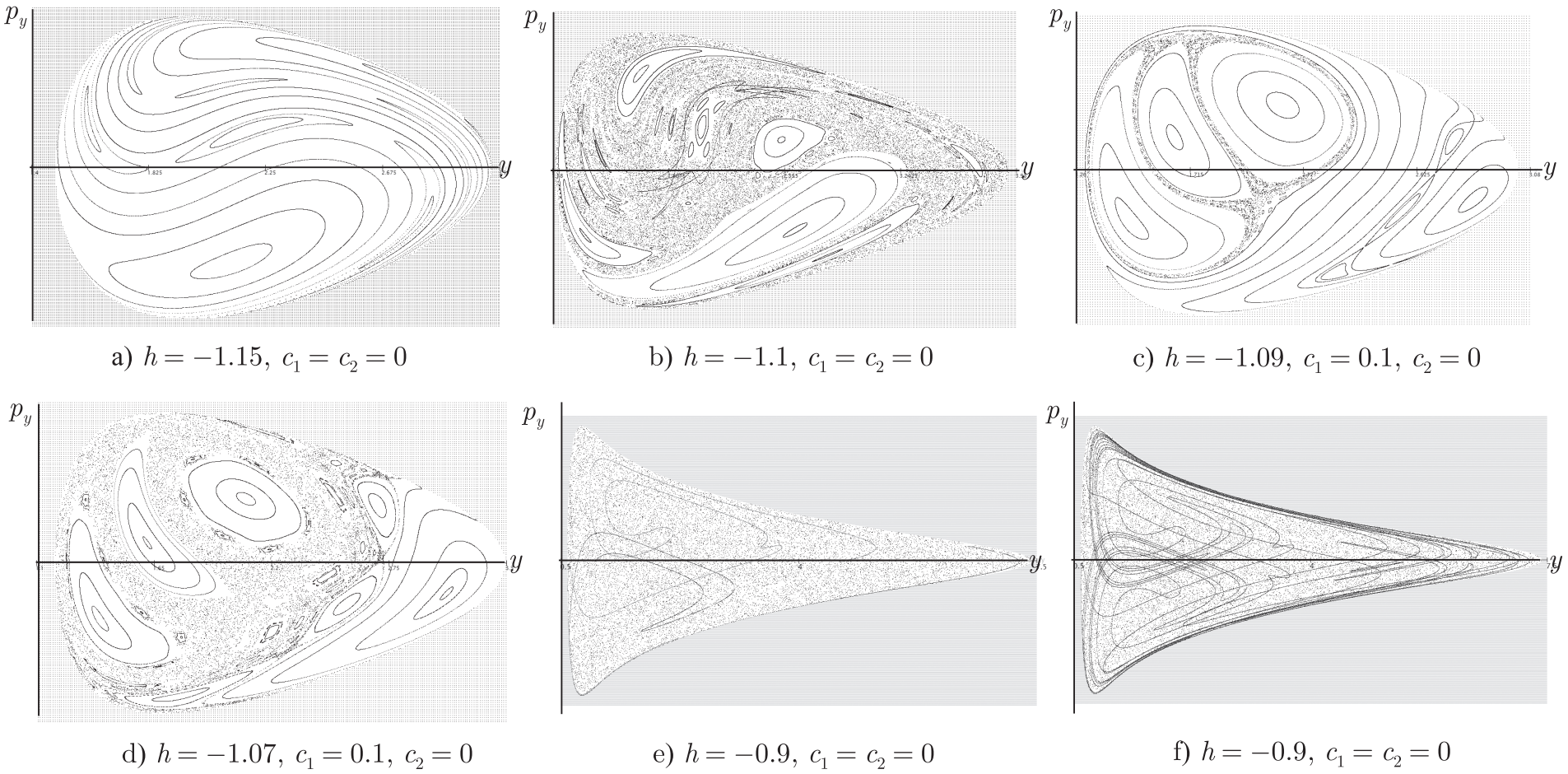

Fig. 2: The Poincaré map of system (57). For all panels, $c_0 = 1, \varepsilon = 0,6$; for the map, the planes $x = 1$ (a–d) and $x = 0.1$ (e ,f) are chosen.

Equations of motion of a gas cloud with a linear velocity field

Now consider in a similar way the case where the motion of a *compressible fluid* (*gas*) is also defined by a linear transformation of the Lagrangian coordinates

$$\boldsymbol{x}(\boldsymbol{a}, t) = \mathbf{F}(t)\boldsymbol{a}; \tag{58}$$

for the compressible medium, the condition $\det \mathbf{F} = 1$ is obviously not valid. Clearly, the velocity field is linear in the coordinates of the fluid particles:

$$\boldsymbol{v}(\boldsymbol{x}, t) = \dot{\boldsymbol{x}} = \dot{\mathbf{F}}(t)\mathbf{F}^{-1}(t)\boldsymbol{x}.$$

In this case, the equations that describe the flow in the given case (for potential forces) have the following Lagrangian representation:

$$\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^T \ddot{\boldsymbol{x}} = -\frac{\partial U}{\partial \boldsymbol{a}} - \frac{1}{\rho}\frac{\partial p}{\partial \boldsymbol{a}}, \tag{59}$$

and the continuity equation in the Lagrangian form is

$$\dot{\rho} + \rho \operatorname{Tr}\left(\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^{-1} \frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{a}}\right) = 0; \tag{60}$$

here, $\operatorname{Tr}\left(\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^{-1} \frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{a}}\right) = \operatorname{div} \boldsymbol{v}(\boldsymbol{x}, t)$.

For the flow of the structure under study (58), the continuity equations can easily be integrated. Indeed, if we introduce the notation $\varphi(t) = \det \mathbf{F}(t)$ using the relationship

$\left(\frac{\partial\varphi}{\partial\mathbf{F}}\right)^T = \varphi\mathbf{F}^{-1}$, we find that $\mathrm{Tr}\left(\left(\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{a}}\right)^{-1}\frac{\partial\dot{\boldsymbol{x}}}{\partial\boldsymbol{a}}\right) = \frac{\dot{\varphi}}{\varphi}$; therefore,

$$\rho(\boldsymbol{a},t) = \frac{f(\boldsymbol{a})}{\varphi(t)}, \tag{61}$$

where the function $f(\boldsymbol{a})$ is time-independent.
Except the four functions $\boldsymbol{x}(\boldsymbol{a},t), p(\boldsymbol{a},t)$, the medium at hand is described by three additional scalar quantities:
the density $\rho(\boldsymbol{a},t)$,
the specific internal energy $U_{\mathrm{in}}(\boldsymbol{a},t)$,
the temperature $T(a,t)$.
Therefore, it is necessary to complement the system (59), (60) with three other equations. Depending on these assumptions, various gasdynamic models can be obtained. Consider three of them that are most widely known, emphasizing the explicit assumptions. Unless the opposite is stipulated, we assume in what follows that the potential of the external forces applied to the system, U, is zero.

The Ovsyannikov Model [23]

1°. *The gas is ideal and can be described by the equation of state*

$$p = \rho RT, \tag{62}$$

where R is the universal gas constant.

2°. *The gas is polytropic, and its internal energy depends linearly on the temperature,*

$$U_{\mathrm{in}} = c_V T, \tag{63}$$

where $c_V = \mathrm{const}$ is the specific heat at constant volume.

3°. *The gas flow is adiabatic (i. e., there is no heat exchange between different parts of the gas volume); therefore, the energy variations are described by the equation*

$$\dot{U}_{\text{in}} = -p\left(\frac{1}{\rho}\right)^{\cdot}. \tag{64}$$

REMARK 5. Equation (64) is a consequence of the first principle of thermodynamics, $\delta Q = dU_{\text{in}} + p\,dV$, where, in view of assumption 3°, it is necessary to set $\delta Q = 0$, ($V = \frac{1}{\rho}$).

REMARK 6. Recall that, due to the well-known thermodynamic identity

$$\left(\frac{dU_{\text{in}}}{dV}\right)_T = \left(T\left(\frac{\partial p}{\partial T}\right)_V - p\right)$$

the internal energy of the ideal gas (62) depends only on the temperature, $U_{\text{in}} = U_{\text{in}}(T)$.

We find using equations (62)–(64) and taking into account the relationship (61) that

$$\frac{\dot{p}}{p} + \gamma\frac{\dot{\varphi}}{\varphi} = 0,$$

where the dimensionless constant $\gamma = 1 + \frac{R}{c_V}$ is the adiabatic index. Thus, for the thermodynamic quantities, we have

$$p(\boldsymbol{a}, t) = \frac{g(\boldsymbol{a})}{\varphi^{\gamma}(t)}, \quad U_{\text{in}}(\boldsymbol{a}, t) = \frac{1}{\gamma - 1}RT(\boldsymbol{a}, t) = \frac{1}{\gamma - 1}\varphi^{1-\gamma}(t)\frac{g(\boldsymbol{a})}{f(\boldsymbol{a})},$$

where $g(\boldsymbol{a})$ is an arbitrary, time-independent quantity.
Thus, for the existence of a solution of the form (58) in this case, it should necessarily be required that

$$\frac{1}{f(\boldsymbol{a})}\nabla_a g(\boldsymbol{a}) = \mathbf{V}\boldsymbol{a}, \tag{65}$$

where $\mathbf{V}$ is a certain constant matrix, $\boldsymbol{\nabla}_a = \left(\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3}\right)$. Then the equations of motion for $\mathbf{F}(t)$ are

$$\mathbf{F}^T\ddot{\mathbf{F}} + (\det \mathbf{F})^{1-\gamma}\mathbf{V} = 0 \qquad (\textit{the Ovsyannikov equations}). \tag{66}$$

As Ovsyannikov has shown [23], it is sufficient to consider the following solutions of equation (65).

Theorem 3. *Any solution of equation* (65) *reduces via a linear transformation of the Lagrangian coordinates to one of the following four types* (*depending on the rank of the matrix* $\mathbf{V}$)*:*

(I) $\mathbf{V} = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\varepsilon_i = \pm 1 \quad (i = 1, 2, 3)$,
$g(\boldsymbol{a}) = g(s), \quad f(\boldsymbol{a}) = 2g'(s), \quad s = (\boldsymbol{a}, \mathbf{V}\boldsymbol{a})$;

(II) $\mathbf{V} = \begin{Vmatrix} \varepsilon_1 & \delta & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad \varepsilon_i = \pm 1 \quad (i = 1, 2)$,

if $\delta \neq 0$ *then* $g(\boldsymbol{a}) = g(s), \quad f(\boldsymbol{a}) = \frac{a_1 g'(s)}{(\boldsymbol{a}, \mathbf{V}\boldsymbol{a})}, \quad s = a_1 \bar{s}\left(\frac{a_2}{a_1}\right), \quad \ln \bar{s}(\lambda) = \int \frac{\varepsilon_2 \lambda \, d\lambda}{\varepsilon_1 + \delta\lambda + \varepsilon_2 \lambda}$;

if $\delta = 0$ *then* $g(\boldsymbol{a}) = g(s), \quad f(\boldsymbol{a}) = 2g'(s), \quad s = (\boldsymbol{a}, \mathbf{V}\boldsymbol{a})$;

(III) $\mathbf{V} = \mathrm{diag}(\varepsilon, 0, 0)$, $\varepsilon = \pm 1, \quad g(\boldsymbol{a}) = g(a_1), \quad f(\boldsymbol{a}) = \frac{\varepsilon}{a_1} g'(a_1)$;

(IV) $\mathbf{V} = 0, \quad g(\boldsymbol{a}) = \mathrm{const}, \quad f(\boldsymbol{a})$ *is an arbitrary function.*

From the physical standpoint, case I with a sign-definite matrix $\mathbf{V}$ is most interesting. In particular, if we set $\mathbf{V} = \mathrm{diag}(-1, -1, -1)$ (i. e., $s = -(\boldsymbol{a}, \boldsymbol{a})$) and choose a linear function $g(s)$, we will obtain

$$g(\boldsymbol{a}) = \frac{1}{2}\rho_0(d_0^2 - (\boldsymbol{a}, \boldsymbol{a})), \quad f(\boldsymbol{a}) = \rho_0, \tag{67}$$

where we must assume that $\rho_0 > 0$ (since the density of the gas is positive). In this case, the gas is distributed with a constant density $\rho = \rho_0 / \det(\mathbf{F})$ inside the finite ellipsoidal volume

$$(\boldsymbol{a}, \boldsymbol{a}) = (\boldsymbol{x}, (\mathbf{F}\mathbf{F}^T)^{-1}\boldsymbol{x}) \leqslant d_0^2. \tag{68}$$

Therefore, the solution of the form (58) in this case remains valid upon adding gravitational forces (the matrix (11) with $\mathbf{A}_0 = \mathbf{E}$ being added to the right-hand side of equation (66)). The problem of the motion of a compressible self-gravitating gas cloud was formulated in [29].

The Dyson Model [26]

Assumptions 1° *and* 3° *coincide with those in the preceding case, whereas, instead of the polytropic behavior, we assume that*

2°. *The gas is isothermal at the initial time, i. e.,* $T(\boldsymbol{a}, t = 0)$ *does not depend on* $\boldsymbol{a}$.

We substitute the pressure from the equation of state (62) into (64) and make use of (61) to obtain

$$\frac{\dot{U}_{\text{in}}}{RT} = -\frac{\dot{\varphi}}{\varphi}. \tag{69}$$

At the same time, as mentioned above (see Note 9), the internal energy depends only on the temperature, and the right-hand side of (69) does not depend on $\boldsymbol{a}$; therefore, the gas remains isothermal at all later times and (69) can be represented as

$$\varphi \frac{dU_{\text{in}}}{d\varphi} + RT = 0. \tag{70}$$

By integrating this equation in view of (63), we obtain a relationship between T and φ:

$$\varphi = \varphi_0 \exp\left(-\int (RT)^{-1} \left(\frac{dU_{\text{in}}}{dT}\right) dT\right). \tag{71}$$

Thus, according to (61), (62), and (70), the pressure can ultimately be written as

$$p(\boldsymbol{a}, t) = \frac{RT(\varphi(t))}{\varphi(t)} f(\boldsymbol{a}). \tag{72}$$

We substitute (72) into (59) and restrict ourselves to the case where no external forces are present (i. e., $U = 0$). Thus, we find that the existence of a solution of the form (58) requires that $\ln f(\boldsymbol{a})$ be a uniform quadratic function of the Lagrangian coordinates. Since the Lagrangian coordinates are defined to within a nonsingular linear substitution, we can represent $f(\boldsymbol{a})$ in the form

$$f(\boldsymbol{a}) = \frac{m}{(2\pi)^{3/2}} \exp(-\frac{1}{2}(\boldsymbol{a}, \boldsymbol{a})), \tag{73}$$

where $m = \int \rho(\boldsymbol{x}) d^3\boldsymbol{x} = \int f(\boldsymbol{a}) d^3\boldsymbol{a}$ is the mass of the gas.

Finally, for the elements of the matrix $\mathbf{F}$, we obtain the equation of motion

$$\mathbf{F}^T \ddot{\mathbf{F}} = RT(\varphi)\mathbf{E} \qquad (\textit{the Dyson equations}). \tag{74}$$

Model of a Cooling Gas Cloud (Fujimoto [27])

In this model, the assumptions 1° *and* 2° *coincide with those in Ovsyannikov's case, i. e., the gas is assumed to be ideal and polytropic, while the third assumption in this case has the form*

3°. *The motion of the gas is not adiabatic, the variations in the internal energy satisfying the equation*

$$\rho \dot{U}_{\text{in}} + p \operatorname{Tr}\left(\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} \right)^{-1} \frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{a}} \right) = -æ\rho^n T^m. \tag{75}$$

REMARK 7. Equations (75) differ from the equations of an adiabatic process (64) by the terms $-æ\rho^n T^m$.

We use equations (61)–(63) to eliminate U_{in} from equation (75) and find

$$\frac{\dot{p}}{p} + \gamma \frac{\dot{\varphi}}{\varphi} = -\frac{æ(\gamma - 1)}{R} \rho^{n-1} T^{m-1}. \tag{76}$$

To obtain a solution in the form (58), we additionally require that

$$m = 1, \quad \rho(\boldsymbol{a}, t) = \frac{\rho_0}{\varphi(t)},$$

where $\rho_0 = \mathrm{const}$ is independent of $\boldsymbol{a}$, i. e., the density is constant inside the cloud. The solution of equation (76) in this case has the form $p(\boldsymbol{a}, t) = \sigma(t) g(\boldsymbol{a})$, where $\sigma(t)$ satisfies the equation

$$\frac{\dot{\sigma}}{\sigma} + \gamma \frac{\dot{\varphi}}{\varphi} = -\bar{æ}\varphi^{1-n}, \quad \bar{æ} = \frac{æ(\gamma - 1)}{R} \rho_0^{n-1}. \tag{77}$$

The function $g(\boldsymbol{a})$ should obviously satisfy equation (65), and it can easily be shown that, according to Theorem 3, we may choose

$$g(\boldsymbol{a}) = 1 - (\boldsymbol{a}, \boldsymbol{a}), \quad f(\boldsymbol{a}) = \mathrm{const}.$$

The condition of the finiteness of the total gas mass implies that $\mathbf{V} = \mathrm{diag}(-1, -1, -1)$, the gas occupying initially the region $(\boldsymbol{a}, \boldsymbol{a}) \leqslant 1$ (in the original physical variables, this inequality specifies an ellipsoid of the form $(\boldsymbol{x}, (\mathbf{F}, \mathbf{F})^{-1}|_{t=0}\, \boldsymbol{x}) \leqslant 1$). Finally, we obtain the system of equations describing the dynamics of the cooling cloud in the form

$$\mathbf{F}^T \ddot{\mathbf{F}} = \frac{2\sigma\varphi}{\rho_0} \mathbf{E} \left(+2\varepsilon \int_0^\infty \mathbf{F}^T (\mathbf{F}\mathbf{F}^T + \lambda \mathbf{E})^{-1} \mathbf{F} \frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{F}^T + \lambda \mathbf{E})}} \right),$$

$$(\ln(\sigma\varphi^\gamma))^{\cdot} = -\bar{æ}\varphi^{1-n}.$$

The parenthesized term describes the gravitational interaction between the particles of the cloud. Allowances for the gravitational interaction in the solution of the form (59) are possible in this case due to the uniformity of gas in the cloud ($\rho_0 = \mathrm{const}$).

Model of a Dust Cloud (Gravitational Collapse)

1°. *The medium (dust) does not counteract deformations,*

$$p \equiv 0.$$

2°. *At the initial time, the particles are distributed uniformly (inside the ellipsoid),*

$$\rho(t, \boldsymbol{a})\big|_{t=0} = \rho_0 = \mathrm{const}.$$

For a solution of the form (59), the density obviously does not depend on the coordinates at all subsequent times, being determined by the relationship

$$\rho(t) = \frac{\rho_0}{\det \mathbf{F}(t)}.$$

Therefore, allowances for the gravitational attraction of particles in the clouds are possible in this model in the framework of the linear solution (59), and the equations of motion can be written as

$$\mathbf{F}^T\ddot{\mathbf{F}} = 2\varepsilon \int\limits_0^\infty \mathbf{F}^T(\mathbf{F}\mathbf{F}^T + \lambda\mathbf{E})^{-1}\mathbf{F}\frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{F}^T + \lambda\mathbf{E})}}. \tag{78}$$

This model is used in astrophysics to describe the gravitational collapse [25]. In particular, it is applied in [24] to the description of the collapse of an elliptic has cloud at zero temperature.

Lagrangian formalism, symmetries, and first integrals

We will now show that the Dyson equations (74), the Ovsyannikov equations (66) under the condition (67), and the equations of a dust cloud (78) admit a natural Lagrangian description. It can be shown by means of direct calculations that the equations of motion can be written in the form

$$\left(\frac{\partial L}{\partial \dot{\mathbf{F}}}\right)^{\cdot} - \frac{\partial L}{\partial \mathbf{F}} = 0,$$
$$L = \frac{1}{2}\,\mathrm{Tr}(\dot{\mathbf{F}}\dot{\mathbf{F}}^T) - U_g(\mathbf{F}), \tag{79}$$

where

$$U_g(\mathbf{F}) = \begin{cases} U_{in}(\varphi) \textit{ for the Dyson model,} \\ \dfrac{1}{\gamma-1}\varphi^{1-\gamma} - 2\varepsilon \displaystyle\int_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{F}^T + \lambda\mathbf{E})}} \textit{ for the Ovsyannikov model with gravitation,} \\ -2\varepsilon \displaystyle\int_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{F}^T + \lambda\mathbf{E})}} \textit{ for the dust-cloud model,} \end{cases} \tag{80}$$

where, as above, $\varphi = \det \mathbf{F}, \mathbf{F} \in GL(3)$.

REMARK 8. In the Dyson model, the Lagrangian representation (79) can be directly obtained from the Hamiltonian principle for barotropic flows (see [6])

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = \delta \int_{t_1}^{t_2} W\, dt, \tag{81}$$

where T and U are the kinetic and the potential energy of the fluid and W is the barotropic potential that satisfies the equation

$$\delta W = \int p \frac{\delta \rho}{\rho} d^3 \boldsymbol{x}. \tag{82}$$

Based on the above assumption, we obtain for our case within a constant:

$$W = \int RT \ln \rho d^3 \boldsymbol{x} = U_{\text{in}}. \tag{83}$$

These considerations can also be generalized to the Ovsyannikov model.

By analogy with the fluid ellipsoid (see Section 2, § 4), we conclude that the system (79) is invariant with respect to linear transformations of the form

$$\mathbf{F}' = \mathbf{S}_1 \mathbf{F} \mathbf{S}_2, \quad \mathbf{S}_1, \mathbf{S}_2 \in SO(3), \tag{84}$$

which form a symmetry group $\Gamma = SO(3) \otimes SO(3)$.

The Dedekind reciprocity law (Teorem 1 in Part 1), which corresponds to a discrete transformation $\mathbf{F}' = \mathbf{F}^T$, is also valid in the dynamics of gas clouds.

According to the Noether second theorem, integrals of motion linear in velocity — the vorticity and total angular momentum of the system – correspond to the transformations (84) and can be represented in the matrix form

$$\boldsymbol{\Xi} = \mathbf{F}^T \dot{\mathbf{F}} - \dot{\mathbf{F}}^T \mathbf{F}, \quad \mathbf{M} = \mathbf{F} \dot{\mathbf{F}}^T - \dot{\mathbf{F}} \mathbf{F}^T. \tag{85}$$

In addition, there is also a quadratic integral, the total energy of the system

$$\mathcal{E} = \frac{1}{2} \operatorname{Tr}(\dot{\mathbf{F}} \dot{\mathbf{F}}^T) + U_g(\mathbf{F}). \tag{86}$$

Symmetry-based reduction and hamiltonian formalism

It is not difficult to carry out a reduction based on the linear integrals (85) using the results of the preceding section. To this end, we make use of the Riemannian decomposition

$$\mathbf{F} = \mathbf{Q}^T \mathbf{A} \boldsymbol{\Theta}, \quad \mathbf{Q}, \boldsymbol{\Theta} \in SO(3), \quad \mathbf{A} = \mathrm{diag}(A_1, A_2, A_3).$$

For the Lagrangian function of the gas cloud (79), in view of the equations

$$\dot{\mathbf{Q}} = \mathbf{w}\mathbf{Q}, \quad \dot{\boldsymbol{\Theta}} = \boldsymbol{\omega}\boldsymbol{\Theta},$$

we obtain the expression

$$L = \frac{1}{2}\sum \dot{A}_i^2 + \frac{1}{4}\sum (A_j + A_k)^2 (\mathrm{w}_i - \omega_i)^2 + (A_j - A_k)^2 (\mathrm{w}_i + \omega_i)^2 - U_g(\mathbf{A}).$$

We denote the three-dimensional vector of semiaxes as $\boldsymbol{q} = (A_1, A_2, A_3)$ and represent the equations of motion in the form

$$\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)^{\cdot} - \frac{\partial L}{\partial \boldsymbol{q}} = 0,$$
$$\left(\frac{\partial L}{\partial \mathrm{w}}\right)^{\cdot} = \frac{\partial L}{\partial \mathrm{w}} \times \mathrm{w}, \quad \left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right)^{\cdot} = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega}.$$

This is an analog of the Riemann equations (35), (34) for the case of a gas cloud (the difference is in the absence of the term containing pressure). These equations can easily be written in a matrix form similar to (35) [26].

The Lagrangian transformation

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}, \quad \boldsymbol{m} = \frac{\partial L}{\partial \dot{\mathrm{w}}}, \quad \boldsymbol{\mu} = \frac{\partial L}{\partial \dot{\boldsymbol{\omega}}}$$

yields a Hamiltonian system

$$\begin{gathered} \boldsymbol{q} = \frac{\partial H}{\partial \dot{\boldsymbol{p}}}, \quad \boldsymbol{p} = -\frac{\partial H}{\partial \dot{\boldsymbol{q}}}, \quad \dot{\boldsymbol{m}} = \boldsymbol{m} \times \frac{\partial H}{\partial \boldsymbol{m}}, \quad \dot{\boldsymbol{\mu}} = \boldsymbol{\mu} \times \frac{\partial H}{\partial \boldsymbol{\mu}}, \\ H = \frac{1}{2}\sum p_i^2 + \frac{1}{4}\sum \left(\frac{m_i + \mu_i}{q_j - q_k}\right)^2 + \left(\frac{m_i - \mu_i}{q_j + q_k}\right)^2 + U_g(\boldsymbol{q}). \end{gathered} \tag{87}$$

The Poissonian structure of the system (87) has the form

$$\{q_i, p_j\} = \delta_{ij}, \quad \{m_i, m_j\} = \varepsilon_{ijk} m_k, \quad \{\mu_i, \mu_j\} = \varepsilon_{ijk}\mu_k, \tag{88}$$

where the zero brackets are omitted. As above, the bracket (88) has two Casimir functions

$$\Phi_m = (\boldsymbol{m}, \boldsymbol{m}), \quad \Phi_\mu = (\boldsymbol{\mu}, \boldsymbol{\mu}),$$

which correspond to the squared total momentum and the vorticity of the system.
In the general case ($\Phi_m \neq 0$, $\Phi_\mu \neq 0$)*, we have a Hamiltonian system with five degrees of freedom.*
In the particular case of $\Phi_m = 0$ *or* $\Phi_\mu = 0$*, we have a system with four degrees of freedom.*
If $\Phi_m = \Phi_\mu = 0$*, we obtain a system with three degrees of freedom* similar to the problem of the motion of a unit-mass point in $\mathbb{R}^3 = \{\boldsymbol{q}\}$.

Particular cases of motion

Case of $\gamma = \frac{5}{3}$ (Monoatomic Gas)

Consider, in greater detail, the case of the expansion of an ellipsoidal cloud of ideal monoatomic gas in the absence of gravitation; we will show that the system has additional symmetries in this case, where, as is known, $c_V = \frac{3}{2}R$ and, therefore, $\gamma = \frac{5}{3}$.

We use (79) to represent the Lagrangian of the system as

$$L = \frac{1}{2}\,\mathrm{Tr}(\dot{\mathbf{F}}\dot{\mathbf{F}}^T) - U_g(\mathbf{F}), \quad U_g(\mathbf{F}) = \frac{3}{2}k\frac{1}{(\det \mathbf{F})^{2/3}}, \tag{89}$$

where $k = \mathrm{const}$ is a positive constant (introduced for convenience). The integrals — vorticity $\mathbf{\Xi}$, momentum $\mathbf{M}$, and energy $\mathcal{E}$ — were mentioned above (85), (86).

We denote the eigenvalues of the matrices $\mathbf{F}\mathbf{F}^T$ as A_1^2, A_2^2, A_3^2 and call A_i the principal semiaxes of th gas ellipsoid (A_i coincides with the semiaxes of the gas ellipsoid in Ovsyannikov's model at the pressure and density distribution (67); for Dyson's model with a normal density distribution (73), this term is only conventional). We define an analog of the central moment of inertia of the system by the formula

$$I = \mathrm{Tr}\,\mathbf{F}\mathbf{F}^T = A_1^2 + A_2^2 + A_3^2. \tag{90}$$

As we can see, according to (89), the dynamics of the could can be described in this case by a natural Lagrangian system with a uniform potential of uniformity degree $\alpha = -2$ (for an arbitrary γ, the uniformity degree is $\alpha = 3(1-\gamma)$). We use the Lagrange–Jacobi formula for uniform systems [30] to obtain

$$\ddot{I} = 4\mathcal{E} = \mathrm{const},$$

where $\mathcal{E}$ is the energy of the system (for an arbitrary γ, we find $\ddot{I} = 4\mathcal{E} - 2(3(1-\gamma)+2)U_g$).

The integration of this relationship yields

$$I = 2\mathcal{E}t^2 + at + b, \tag{91}$$

where the integration constants a and b can be expressed in terms of the phase variables and time according to the formulas

$$a = 2\,\mathrm{Tr}(\mathbf{F}^T\dot{\mathbf{F}}) - 4\mathcal{E}t, \quad b = 2\mathcal{E}t^2 - 2\,\mathrm{Tr}(\mathbf{F}^T\dot{\mathbf{F}})t + I. \tag{92}$$

In fact, a and b are nonautonomic (explicitly time-dependent) integrals of the system considered.

For the first time, the integral (91) for uniform systems of a degree of -2 was noted by Jacobi in the problem of the motion of particles in a straight line. For the problem of the motion of a gas cloud, the Jacobi integral was found in [28]. The integrals (92) for system (89) were indicated in [31], while corresponding symmetries in the particular case of $\Xi = 0$ were mentioned in [32].

Proposition 1. *At $t \to \pm\infty$, at least one of the semiaxes, A_i, of the gas cloud goes to infinity.*

Except the nonautonomic integrals (92), the systems in this case admits an autonomic quadratic integral independent of the energy integral,

$$J = 2I\mathcal{E} - [\mathrm{Tr}(\mathbf{F}^T\dot{\mathbf{F}})]^2. \tag{93}$$

For uniform systems of degree -2, this integral was found in a more general case in [34]. For the system (89) in the particular case of $\Xi = 0$, it is also given in [32]. Preliminary results on symmetries for this integral were given in [35, 36, 37].

For uniform natural systems of degree -2, a special reduction can be made to lower the number of degrees of freedom by unity. We describe it in application to the considered system (89).
We carry out a substitution of time and a (projective) substitution of variables

$$dt = I\,d\tau, \quad \mathbf{G} = I^{-1/2}\mathbf{F}. \tag{94}$$

It can easily be shown by direct calculation that the evolution of the matrix $\mathbf{G}(t)$ can be described by a Lagrangian system with a constraint φ in the following form:

$$\begin{gathered} L = \frac{1}{2}\operatorname{Tr}\left(\frac{d\mathbf{G}}{d\tau}\frac{d\mathbf{G}^T}{d\tau}\right) - \bar{U}_g(\mathbf{G}), \quad \bar{U}_g(\mathbf{G}) = \frac{3}{2}k\frac{1}{(\det\mathbf{G})^{2/3}}, \\ \varphi = \operatorname{Tr}(\mathbf{G}\mathbf{G}^T) = 1. \end{gathered} \tag{95}$$

A relationship between the "old" time t and the "new" time τ can be found using (91). Note that the system (95) differs from the Dirichlet system, since the constraint φ is different in this case (in the Dirichlet problem, $\det\mathbf{G} = 1$). It is interesting that the energy integral for the system (95) coincides with the integral (93):

$$\bar{\mathcal{E}} = \frac{1}{4}J = \frac{1}{2}\operatorname{Tr}\left(\frac{d\mathbf{G}}{d\tau}\frac{d\mathbf{G}^T}{d\tau}\right) - \bar{U}_g(\mathbf{G}).$$

The linear integrals in the system (95) remains the same,

$$\mathbf{\Xi} = \mathbf{G}^T\frac{d\mathbf{G}}{d\tau} - \frac{d\mathbf{G}^T}{d\tau}\mathbf{G}, \quad \mathbf{M} = \mathbf{G}\frac{d\mathbf{G}^T}{d\tau} - \frac{d\mathbf{G}}{d\tau}\mathbf{G}^T;$$

furthermore, the system (95) is invariant with respect to the same transformations (84), which form a group $\Gamma = SO(3) \otimes SO(3)$. Therefore, a symmetry-based reduction similar to the above-described one is possible (see Part II, Section 3), with the only difference that, in this case, the following relationship between the semiaxes is valid:

$$\bar{A}_1^2 + \bar{A}_2^2 + \bar{A}_3^2 = 1. \tag{96}$$

We use the Riemann decomposition of the matrix $\mathbf{G} = \mathbf{Q}^T \mathbf{A} \boldsymbol{\Theta}$, $\mathbf{Q}, \boldsymbol{\Theta} \in SO(3)$, $\mathbf{A} = \mathrm{diag}(\bar{A}_1, \bar{A}_2, \bar{A}_3)$, to obtain, in this case, a system similar to (87) but with an additional constraint (96). To take this constraint into account and represent the equations in the most symmetric form, we define variables $\boldsymbol{q}$ and $\boldsymbol{K}$ according to the formulas

$$q_i = A_i, \quad \boldsymbol{K} = \boldsymbol{q} \times \frac{d\boldsymbol{q}}{d\tau}. \tag{97}$$

Then we finally obtain a reduced system in the form

$$\begin{gathered}
\frac{d\boldsymbol{K}}{d\tau} = \boldsymbol{K} \times \frac{\partial \bar{H}}{\partial \boldsymbol{K}} + \boldsymbol{q} \times \frac{\partial \bar{H}}{\partial \boldsymbol{q}}, \quad \frac{d\boldsymbol{q}}{d\tau} = \boldsymbol{q} \times \frac{\partial \bar{H}}{\partial \boldsymbol{K}}, \\
\frac{d\boldsymbol{m}}{d\tau} = \boldsymbol{m} \times \frac{\partial \bar{H}}{\partial \boldsymbol{m}}. \quad \frac{d\boldsymbol{\mu}}{d\tau} = \boldsymbol{\mu} \times \frac{\partial \bar{H}}{\partial \boldsymbol{\mu}}, \\
\bar{H} = \frac{1}{2}(\boldsymbol{K}, \boldsymbol{K}) + \frac{1}{4} \sum \left(\frac{m_i + \mu_i}{q_j - q_k} \right)^2 + \left(\frac{m_i - \mu_i}{q_j + q_k} \right)^2 + U_g(\boldsymbol{q}).
\end{gathered} \tag{98}$$

The (nonzero) Poisson brackets corresponding to the system (97) are as follows:

$$\{K_i, K_j\} = \varepsilon_{ijk} K_k, \quad \{K_i, q_j\} = \varepsilon_{ijk} q_k, \quad \{m_i, m_j\} = \varepsilon_{ijk} m_k, \quad \{\mu_i, \mu_j\} = \varepsilon_{ijk} \mu_k.$$

Thuis Poisson structure, as is known, corresponds to the algebra $e(3) \oplus so(3) \oplus so(3)$ and has four Casimir functions,

$$\Phi_K = (\boldsymbol{K}, \boldsymbol{q}), \quad \Phi_q = (\boldsymbol{q}, \boldsymbol{q}),$$
$$\Phi_m = (\boldsymbol{m}, \boldsymbol{m}), \quad \Phi_\mu = (\boldsymbol{\mu}, \boldsymbol{\mu});$$

in view of the definition, (97), of the variables $\boldsymbol{K}$ and $\boldsymbol{q}$, we have in this case

$$\Phi_K = 0, \quad \Phi_q = 1.$$

Thus, we ultimately conclude that

1. *if* $\Phi_m, \Phi_\mu \neq 0$*, equations* (87) *correspond to a Hamiltonian system with four degrees of freedom;*
2. *if* $\Phi_m = 0$ (*or* $\Phi_\mu = 0$)*, we obtain a system with three degrees of freedom;*
3. *if* $\Phi_m = \Phi_\mu = 0$*, we obtain a system with two degrees of freedom.*

Gaffet [33] for the case of $\Phi_\mu = 0$ found two additional first integrals (of the sixth degree in the velocities) independent of the energy integral.
These integrals are polynomials of the sixth degree in momenta and have the form

$$I_6 = 36k^2 \left(Y_0 Y_2 - \frac{1}{4} Y_1^2 + 3X_2 + T(X_0 + Y_0^2) \right)$$
$$+6k(4T^2 Y_0 + 3PY_1 + 6TY_2) + 27P^2 + 4T^3,$$
$$L_6 = \left(\mathbf{A}^2 m, \mathbf{V}_0 \mathbf{A}^2 m \times (\mathbf{V}_0^2 \mathbf{A}^2 m + \frac{3k}{(q_1 q_2 q_3)^{2/3}} m) \right),$$

where $\mathbf{A} = \mathrm{diag}(q_1, q_2, q_3)$ and the quantities X_i, Y_i, P, and T can be expressed in terms of the symmetric matrix

$$\mathbf{V}_0 = \begin{pmatrix} \frac{1}{3}\sum_{i=1}^{3}\frac{K_i}{q_i} - \frac{K_1}{q_1} & \frac{m_3}{q_1^2-q_2^2} & \frac{m_3}{q_3^2-q_1^3} \\ \frac{m_3}{q_1^2-q_2^2} & \frac{1}{3}\sum_{i=1}^{3}\frac{K_i}{q_i} - \frac{K_2}{q_2} & \frac{m_1}{q_2^2-q_3^2} \\ \frac{m_2}{q_3^2-q_1^2} & \frac{m_1}{q_2^2-q_3^2} & \frac{1}{3}\sum_{i=1}^{3}\frac{K_i}{q_i} - \frac{K_3}{q_3} \end{pmatrix}$$

as follows:

$$X_k = (q_1q_2q_3)^{2(k-1)/3}\,\mathrm{Tr}(\mathbf{V}_0^k\mathbf{A}^2), \qquad Y_k = (q_1q_2q_3)^{2(k+1)/3}\,\mathrm{Tr}(\mathbf{V}_0^k\mathbf{A}^{-2}),$$
$$T = -\frac{1}{2}(q_1q_2q_3)^{4/3}\,\mathrm{Tr}(\mathbf{V}_0^2), \qquad P = (q_1q_2q_3)^2 \det \mathbf{V}_0.$$

The Case of Axial Symmetry

As in the case of a fluid ellipsoid, it can easily be shown that the system (79) admits a three-dimensional invariant manifold formed by matrices of the form

$$F = \begin{pmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & w \end{pmatrix}. \tag{99}$$

The liner integrals (85) simplify in this case becoming

$$\Xi_{12} = -M_{12} = 2(u\dot{v} - v\dot{u}), \quad \Xi_{13} = \Xi_{23} = M_{13} = M_{23} = 0. \tag{100}$$

Consider the Ovsyannikov model with gravitation (80) and make the substitution of variables

$$u = \frac{1}{\sqrt{2}} r \cos\psi, \quad v = \frac{1}{\sqrt{2}} r \sin\psi, \quad w = z.$$

Then the Lagrangian function of the system assumes the form

$$\begin{aligned} L &= \frac{1}{2}(\dot{r}^2 + r^2\dot{\psi}^2 + \dot{z}^2) - U_g(r,z), \\ U_g &= \frac{k}{\gamma - 1}\frac{1}{(r^2 z)^{\gamma-1}} + U_e(r,z), \end{aligned} \tag{101}$$

where the energy of the gravitational field U_e can be expressed in terms of elementary functions:

$$U_e = -2\varepsilon \int\limits_0^\infty \frac{d\lambda}{(\lambda + \frac{r^2}{2})\sqrt{\lambda + z^2}} = -\frac{2\varepsilon}{z} \times \begin{cases} \dfrac{2\,\mathrm{arctg}\,\sqrt{\chi^2 - 1}}{\sqrt{\chi^2 - 1}}, & \chi > 1, \\ \dfrac{\ln\left(\frac{1+\sqrt{1-\chi^2}}{1-\sqrt{1-\chi^2}}\right)}{\sqrt{1-\chi^2}}, & \chi < 1, \end{cases}$$

where $\chi = \frac{1}{\sqrt{2}}\frac{r}{z}$ is the semiaxis ratio. Since the Lagrangian (101) is independent of ψ, there is the cyclic integral

$$\frac{\partial L}{\partial \dot{\psi}} = r^2\dot{\psi} = c = \mathrm{const},$$

which coincides with the integrals (100) within a multiplier.

For a fixed value of this integral, we make the Legendre transformation $p_r = \frac{\partial L}{\partial \dot{r}} = \dot{r}, p_z = \frac{\partial L}{\partial \dot{z}} = \dot{z}$ and obtain a Hamiltonian system with two degrees of freedom in the canonical form

$$H = \frac{1}{2}(p_r^2 + p_z^2) + U_*(r, z), \quad U_* = \frac{c^2}{2r^2} + U_g(r, \psi); \tag{102}$$

here, U_* is the reduced potential.
Consider the simplest (integrable) case, the motion of a monoatomic gas ($\gamma = \frac{5}{3}$) without allowances for gravitation (i. e., $U_e = 0$; see also the preceding section).
It was shown above that the system in this case admits a reduction by one more degree of freedom and, therefore, reduces to a quadrature. Indeed, we make a substitution of variables and time of the form

$$r = R\cos\theta, \quad z = R\sin\theta, \quad dt = R^2\, d\tau,$$

where, in view of the conditions $r > 0$ and $z > 0$, the variable $\theta \in (0, \pi/2)$. We obtain the following equations for R and θ:

$$\frac{d^2}{dt^2}(R^2) = 4H = \text{const},$$

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}\frac{c^2}{\cos^2\theta} + \frac{3}{2}\frac{k}{(\cos^2\theta\sin\theta)^{2/3}} = h_1 = \text{const}.$$

The quadrature for θ at $c = 0$, with certain limitations on the initial conditions, was obtained in [28]. As we can see, the evolution of $\theta(t)$ can be determined by the reduced potential

$$\bar{U}_*(\theta) = \frac{1}{2}\frac{c^2}{\cos^2\theta} + \frac{3}{2}\frac{k}{(\cos^2\theta\sin\theta)^{2/3}}.$$

At all values of the parameters c and k, this function has one critical value θ_0 in the interval $(0, \pi/2)$, in which $\bar{U}_*$ reaches its minimum. This value corresponds to the self-similar expansion of a spheroidal gas cloud. In other cases, the expansion of the cloud is accompanied by oscillations in the semiaxis lengths, with θ varying in the interval (θ_1, θ_2), where θ_i are the roots of the equation $\bar{U}_*(\theta) = h_1$.
In the general case, $U_e \neq 0$, the trajectories of the system (102) are not finite. However, it can easily be shown that, at $k > \frac{1}{96} 2^{2/3}(\sqrt{665} - 21)c^2 \approx 0,43c^2$, the reduced potential has a minimum at the point

$$\theta_0 = \operatorname{arctg} \frac{1}{\sqrt{2}}, \quad R_0 = \frac{\sqrt{3}}{8\varepsilon}(c^2 + 3 \cdot 2^{1/3}k).$$

Therefore, near the minimum of the energy $U_*(\theta_0, R_0)$, the trajectories of the system are finite and a Poincaré map can be constructed. Such a map in the plane $\theta = \frac{\pi}{4}$ as the plane of section is shown in Fig. 3. A chaotic layer that originates from the splitting of resonant tori can be clearly seen in this figure, which testifies to the nonintegrability of the system (102).

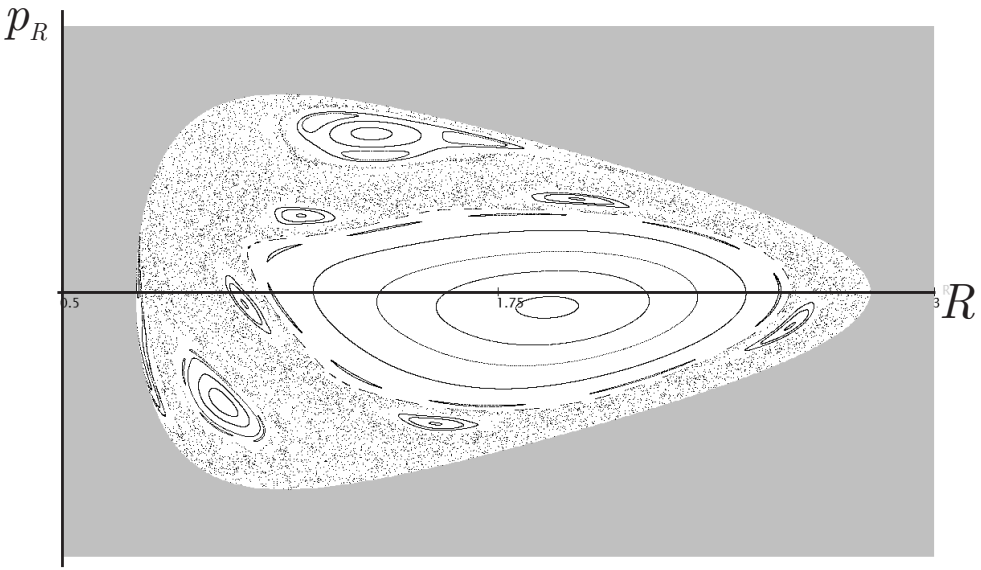

Fig. 3: The Poincaré map of the system (102) at $k = c = \varepsilon = 1$ in the section plane $\theta = \frac{\pi}{4}$.

Generalization of the Riemannian Case

An invariant manifold of the form (48) also exists for gas ellipsoids, i. e.,

$$\mathbf{F} = \begin{Vmatrix} u_1 & v_1 & 0 \\ u_2 & v_2 & 0 \\ 0 & 0 & w_3 \end{Vmatrix}. \tag{103}$$

As in the Riemannian case, it can be shown for a fluid ellipsoid that, in the case of gas, the following relationships are also valid:

$$m_1 = m_2 = \mu_1 = \mu_2 = 0, \quad m_3 = \text{const}, \quad \mu_3 = \text{const}.$$

Thus, we conclude that, according to (87), the evolution of the semiaxes $A_i = q_i, i = 1, 2, 3$ can be described by the third-degree Hamiltonian system

$$H = \frac{1}{2}\boldsymbol{p}^2 + U_*(q), \quad U_* = \frac{c_1^2}{(q_1 - q_2)^2} + \frac{c_2^2}{(q_1 + q_2)^2} + U_g(\boldsymbol{q}), \tag{104}$$

where $\boldsymbol{q}, \boldsymbol{p}$ are canonically conjugate variables and $c_1 = \frac{1}{2}(m_3 + \mu_3)$, $c_2 = \frac{1}{2}(m_3 - \mu_3)$ are fixed constants.

It was shown above that, for a monatomic gas ($\gamma = \frac{5}{3}$), without taking into account gravitation ($U_e = 0$), the system admits a reduction by one more degree of freedom. As a result, we obtain in this case a system of the form

$$\frac{d\boldsymbol{K}}{d\tau} = \boldsymbol{K} \times \frac{\partial \bar{H}}{\partial \boldsymbol{K}} + \boldsymbol{q} \times \frac{\partial \bar{H}}{\partial \boldsymbol{q}}, \quad \frac{d\boldsymbol{q}}{d\tau} = \boldsymbol{q} \times \frac{\partial \bar{H}}{\partial \boldsymbol{K}},$$

$$\bar{H} = \frac{1}{2}\boldsymbol{K}^2 + \bar{U}_*(\boldsymbol{q}), \quad \bar{U}_*(\boldsymbol{q}) = \frac{3}{2}\frac{k}{(q_1 q_2 q_3)^{2/3}} + \frac{c_1^2}{(q_1 - q_2)^2} + \frac{c_2^2}{(q_1 + q_2)^2}.$$

This system is equivalent to the problem of the motion of a spherical top in an axisymmetric potential [18]. As shown in [31], this system is integrable provided $c_1^2 = c_2^2$. At $c_1 = c_2 = 0$, the additional integral of the third degree in the velocities has the form

$$F_3 = K_1 K_2 K_3 - 3k \frac{K_1 q_2 q_3 + K_2 q_3 q_1 + K_3 q_1 q_2}{(q_1 q_2 q_3)^{2/3}}.$$

If $c_1 = c_2 = c \neq 0$, we have an additional sixth-degree integral

$$F_6 = (F_3 + F_a)^2 + 4 \frac{f(q_1^2\theta + 3kq_3^2)(q_2^2\theta + 3kq_3^2)}{q_3^4},$$

where

$$F_a = \frac{4c^2 q_1 q_2 q_3^2}{(q_1^2 - q_2^2)^2} K_3, \quad f = \frac{4c^2 (q_1 q_2 q_3)^{2/3}}{(q_1^2 - q_2^2)^2} q_3^2, \quad \theta = \frac{(q_1 q_2 q_3)^{2/3}}{q_1 q_2} K_1 K_2 - 3k + f.$$

In the more general case of $c_1^2 \neq c_2^2$, the system (103) becomes nonintegrable.
Figure 4 shows the corresponding Poincaré map in the Anduaye variables, which are traditionally used for reductions in the problems of rigid-body motion with a fixed point [18]. The break down of the resonant tori and the birth of isolated periodic solution can be clearly seen from the figure, which is evidence for the nonintegrability of the problem.

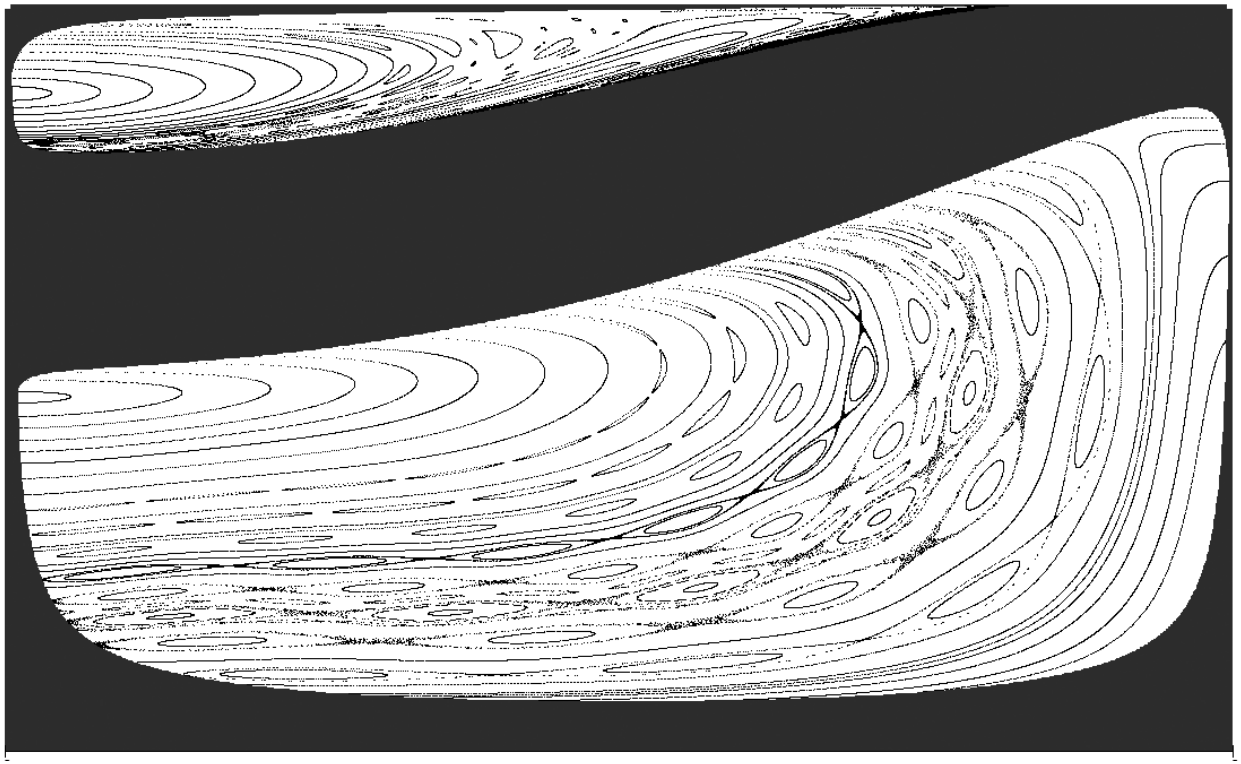

Fig. 4: Poincaré map of the system (103) at an energy level of $\bar{H} = 30$ at $k = 1/3, c_1 = 1, c_2 = 0.3$ in the section plane $g = \pi$.

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