# On the number of isolating integrals in perturbed Hamiltonian systems

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Consider the n-degrees-of-freedom nearly integrable Hamiltonian of the form

$$H = H_0 + \epsilon H_1 \tag{1}$$

where $H_0$ is integrable, i.e, it possesses n-1, in addition to $H_0$, single-valued, independent integrals in involution and suppose that we can define action-angle variables $J_i$, $w_i$ (i=1,...,n), at least in an open domain of phase space. We also suppose that $H_0$ is non-degenerate, i.e.

$$det\left[\frac{\partial^2 H_0}{\partial J_i \partial J_j}\right] \neq 0 \tag{2}$$

On any torus of $H_0$, the unperturbed solution is described by the angle coordinates as follows

$$w_i = \omega_i t + \vartheta_i, \qquad (i = 1, ..., n) \tag{3}$$

where $w_i$ are $mod(2\pi)$ and $\vartheta_i$ arbitrary initial conditions on the torus. We consider the periodic motions of a resonnant torus of $H_0$ with frequency artio

$$\frac{\omega_1}{p_1} = \frac{\omega_2}{p_2} = ... = \frac{\omega_n}{p_n} \tag{4}$$

where $\omega_i = \partial H_0/\partial J_i$ are the frequencies of the integrable part and $p_i$ are non-zero integers with no common divisor. A particular resonant torus is foliated by periodic orbits with period

$$T = \frac{2\pi p_i}{\omega_i} \tag{5}$$

Every periodic orbit is mapped on the quotient manifold of the resonant torus with respect to the foliation induced by the trivially integrable 1-forms $p_{i+1}dw_i - p_1 dw_{i+1} = 0 (i = 1, n-1)$, to a point with coordinates

$$\varphi_i = p_{i+1}\vartheta_1 - p_1\vartheta_{i+1} \quad mod(2\pi s_i), \quad (i = 1, ..., n-1) \tag{6}$$

here $s_i$ are the common divisors of the pairs $p_i, p_{i+1}$.

**theorem 1** *Consider the n-degrees-of-freedom Hamiltonian* $H = H_0 + \epsilon H_1$ *where* $H_0$ *is a non-degenerate integrable part for which action-angle variables can be defined in an open domain of phase space. The quotient manifold* $Q(J_1, ..., J_n)$ *of each resonant torus of* $H_0$ *with respect to the foliation induced by the periodic orbits is the (n-1)-dimensional torus* $T^{n-1}$. *Let* $< H_1 >$ *be the average value of* $H_1$ *along these periodic orbits, which is a scalar function on each* $Q$. *Then*

*a) If in this open domain of phase space, a dense set of resonant tori of* $H_0$ *is found, such that for at least one orbit on each one of them*

$$\frac{\partial}{\partial \varphi_i} < H_1 > \neq 0 \tag{7}$$

*for at least one i,* $H$ *does not possess a complete set of analytic integrals for* $\epsilon \in (-\epsilon_0, \epsilon_0)$ $\{O\}$

*b) If, similarly,*

$$rank \left| \frac{\partial^2 < H_1 >}{\partial \varphi_m \varphi_j} \right| \geq r \qquad (m, j = 1, ..., n-1) \tag{8}$$

*then* $H$ *cannot possess more than (n-r) independant integrals, including* $H$, *which are analytic in the same interval of* $\epsilon$.

*We apply the previous results to the system of three quartic oscillators with a weak coupling described by the Hamiltonian*

$$H = H_0 + \epsilon H_1 = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) + \frac{1}{4}(x_1^4 + x_2^4 + x_3^4) + \epsilon(x_1 x_2 + x_1 x_3) \tag{9}$$

*and prove that* $H$ *does not possess another integral of motion for* $\epsilon \in (-\epsilon_0, \epsilon_0)$ $\{O\}$. *This result, however is valid for all* $\epsilon$, *since equations of motion are invariant with respect to the transformation*

$$x_i \rightarrow c x_i, \quad t \rightarrow .3cmc^{-1}t, \quad \epsilon \quad c^2\epsilon \tag{10}$$

*with arbitrary c.*

*If we select* $H_1 = x_1 x_2$ *we can still prove non-integrability, the Hessian, however, is identically zero and this is due to the existence of a second integral (i.e the energy along the* $x_3$*-axis).*